$$\varphi(t) = \int_{\mathbb{R}} e^{\mathtt{i}tx} \frac{1}{\pi} \frac{dx}{1+x^2}$$

1. $\varphi(t) = \varphi(-t)$.

Fissiamo $t = a > 0$.

$$\frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{\mathtt{i}ax}}{1+x^2}\, dx$$

$$f(z) = \frac{e^{\mathtt{i}az}}{1+z^2}, \qquad z \in \mathbb{C}$$

f è olomorfa in tutto il piano complesso tranne nei punti $\mathtt{i}$ e $-\mathtt{i}$.

$$z(\theta) = \mathtt{i} + \rho e^{\mathtt{i}\theta}$$

$$\dot{z}(\theta) = \mathtt{i}\rho e^{\mathtt{i}\theta}$$

$$f(z(\theta)) = \frac{e^{\mathtt{i}az(\theta)}}{1+z(\theta)^2} = e^{-a} \frac{e^{\mathtt{i}a\rho e^{\mathtt{i}\theta}}}{\rho^2 e^{2\mathtt{i}\theta} + 2\mathtt{i}\rho e^{\mathtt{i}\theta}}$$

$$\int_0^{2\pi} f(z(\theta))\dot{z}(\theta)\, d\theta = e^{-a} \int_0^{2\pi} \frac{e^{\mathtt{i}a\rho e^{\mathtt{i}\theta}}}{2 - \mathtt{i}\rho e^{\mathtt{i}\theta}}\, d\theta$$

Non dipende da ρ. Per $\rho \to 0$

$$\oint f(z)\, dz = e^{-a} \int_0^{2\pi} \frac{e^{\mathtt{i}a\rho e^{\mathtt{i}\theta}}}{2 - \mathtt{i}\rho e^{\mathtt{i}\theta}}\, d\theta \to \pi e^{-a}$$

$$\int_{-R}^{R} \frac{e^{\mathtt{i}ax}}{1+x^2}\, dx + \int_\Gamma f(z)\, dz = \oint f(z)\, dz = \pi e^{-a}$$

$$\left| \int_\Gamma f(z)\, dz \right| = \left| \int_0^\pi f(Re^{\mathtt{i}\theta})\ \mathtt{i}Re^{\mathtt{i}\theta} d\theta \right| = \left| \int_0^\pi \frac{e^{\mathtt{i}aRe^{\mathtt{i}\theta}}}{1+R^2e^{2\mathtt{i}\theta}}\ \mathtt{i}Re^{\mathtt{i}\theta} d\theta \right|$$

$$\left| \int_\Gamma f(z)\, dz \right| \le R \int_0^\pi \frac{e^{-aR\sin\theta}}{|1+R^2e^{2\mathtt{i}\theta}|}\, d\theta$$

$$e^{-aR\sin\theta} \le 1, \qquad |1+R^2e^{2\mathtt{i}\theta}| \ge R^2 - 1$$

$$\left| \int_\Gamma f(z)\, dz \right| \le \frac{R}{R^2-1}\, \pi \to 0 \qquad \text{per } R \to \infty$$

Quindi

$$\varphi(t) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{e^{\mathtt{i}tx}}{1+x^2}\, dx = e^{-|t|}$$