Real and Complex Analysis

existence of radial limits of bounded harmonic functions. The theorems of Plancherel and Cauchy combined give a theorem of Paley and Wiener which, in turn, is used in the Denjoy-Carleman theorem about infinitely differentiable functions on the real line. The maximum modulus theorem gives information about linear transformations on L^p-spaces.

Since most of the results presented here are quite classical (the novelty lies in the arrangement, and some of the proofs are new), I have not attempted to document the source of every item. References are gathered at the end, in Notes and Comments. They are not always to the original sources, but more often to more recent works where further references can be found. In no case does the absence of a reference imply any claim to originality on my part.

The prerequisite for this book is a good course in advanced calculus (set-theoretic manipulations, metric spaces, uniform continuity, and uniform convergence). The first seven chapters of my earlier book "Principles of Mathematical Analysis" furnish sufficient preparation.

Chapters 1 to 8 and 10 to 15 should be taken up in the order in which they are presented. Chapter 9 is not referred to again until Chapter 19. The last five chapters are quite independent of each other, and probably not all of them should be taken up in any one year. There are over 350 problems, some quite easy, some more challenging. About half of these have been assigned to my classes at various times.

The students' response to this course has been most gratifying, and I have profited much from some of their comments. Notes taken by Aaron Strauss and Stephen Fisher helped me greatly in the writing of the final manuscript. The text contains a number of improvements which were suggested by Howard Conner, Simon Hellerstein, Marvin Knopp, and E. L. Stout. It is a pleasure to express my sincere thanks to them for their generous assistance.

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some real y. Hence $w = e^{x+iy}$. This proves (g) and completes the theorem.

We shall encounter the integral of $(1 + x^2)^{-1}$ over the real line. To evaluate it, put $\varphi(t) = \sin t/\cos t$ in $(-\pi/2,\pi/2)$. By (6), $\varphi' = 1 + \varphi^2$. Hence φ is a monotonically increasing mapping of $(-\pi/2,\pi/2)$ onto $(-\infty,\infty)$, and we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\pi/2}^{\pi/2} \frac{\varphi'(t)\,dt}{1+\varphi^2(t)} = \int_{-\pi/2}^{\pi/2} dt = \pi.$$

1

Abstract Integration

Toward the end of the nineteenth century it became clear to many mathematicians that the Riemann integral (about which one learns in calculus courses) should be replaced by some other type of integral, more general and more flexible, better suited for dealing with limit processes. Among the attempts made in this direction, the most notable ones were due to Jordan, Borel, W. H. Young, and Lebesgue. It was Lebesgue's construction which turned out to be the most successful.

In brief outline, here is the main idea: The Riemann integral of a function f over an interval $[a,b]$ can be approximated by sums of the form

$$\sum_{i=1}^{n} f(t_i)m(E_i)$$

where $E_1, \ldots, E_n$ are disjoint intervals whose union is $[a,b]$, $m(E_i)$ denotes the length of E_i, and $t_i \in E_i$ for $n = 1, \ldots, n$. Lebesgue discovered that a completely satisfactory theory of integration results if the sets E_i in the above sum are allowed to belong to a larger class of subsets of the line, the so-called "measurable sets," and if the class of functions under consideration is enlarged to what he called "measurable functions." The crucial set-theoretic properties involved are the following: The union and the intersection of any countable family of measurable sets are measurable; so is the complement of every measurable set; and, most important, the notion of "length" (now called "measure") can be extended to them in such a way that

$$m(E_1 \cup E_2 \cup E_3 \cup \cdots) = m(E_1) + m(E_2) + m(E_3) + \cdots$$

for every countable collection $\{E_i\}$ of pairwise disjoint measurable sets. This property of m is called *countable additivity*.

The passage from Riemann's theory of integration to that of Lebesgue is a process of completion (in a sense which will appear more precisely later). It is of the same fundamental importance in analysis as is the construction of the real number system from the rationals.

The above-mentioned measure m is of course intimately related to the geometry of the real line. In this chapter we shall present an abstract (axiomatic) version of the Lebesgue integral, relative to *any* countably additive measure on *any* set. (The precise definitions follow.) This abstract theory is not in any way more difficult than the special case of the real line; it shows that a large part of integration theory is independent of any geometry (or topology) of the underlying space; and, of course, it gives us a tool of much wider applicability. The existence of a large class of measures, among them that of Lebesgue, will be established in Chap. 2.

Set-theoretic Notations and Terminology

1.1 Some sets can be described by listing their members. Thus $\{x_1, \ldots, x_n\}$ is the set whose members are $x_1, \ldots, x_n$; and $\{x\}$ is the set whose only member is x. More often, sets are described by properties. We write

$$\{x: P\}$$

for the set of all elements x which have the property P. The symbol $\varnothing$ denotes the empty set. The words *collection, family*, and *class* will be used synonymously with *set*.

We write $x \in A$ if x is a member of the set A; otherwise $x \notin A$. If B is a subset of A, i.e., if $x \in B$ implies $x \in A$, we write $B \subset A$. If $B \subset A$ and $A \subset B$, then $A = B$. If $B \subset A$ and $A \neq B$, B is a *proper* subset of A. Note that $\varnothing \subset A$ for every set A.

$A \cup B$ and $A \cap B$ are the union and intersection of A and B, respectively. If $\{A_\alpha\}$ is a collection of sets, where α runs through some index set I, we write

$$\bigcup_{\alpha \in I} A_\alpha \quad \text{and} \quad \bigcap_{\alpha \in I} A_\alpha$$

for the union and intersection of $\{A_\alpha\}$:

$$\bigcup_{\alpha \in I} A_\alpha = \{x: x \in A_\alpha \text{ for at least one } \alpha \in I\}$$
$$\bigcap_{\alpha \in I} A_\alpha = \{x: x \in A_\alpha \text{ for every } \alpha \in I\}.$$

If I is the set of all positive integers, the customary notations are

$$\bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n.$$

If no two members of $\{A_\alpha\}$ have an element in common, then $\{A_\alpha\}$ is a *disjoint collection* of sets.

We write $A - B = \{x: x \in A, x \notin B\}$, and denote the complement of A by A^c whenever it is clear from the context with respect to which larger set the complement is taken.

The *cartesian product* $A_1 \times \cdots \times A_n$ of the sets $A_1, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, \ldots, a_n)$ where $a_i \in A_i$ for $1 = 1, \ldots, n$.

The *real line* (or real number system) is R^1, and

$$R^k = R^1 \times \cdots \times R^1 \qquad (k \text{ factors}).$$

The *extended real number system* is R^1 with two symbols, ∞ and $-\infty$, adjoined, and with the obvious ordering. If $-\infty \le a \le b \le \infty$, the *interval* $[a,b]$ and the *segment* (a,b) are defined to be

$$[a,b] = \{x: a \le x \le b\}, \qquad (a,b) = \{x: a < x < b\}.$$

We also write

$$[a,b) = \{x: a \le x < b\}, \qquad (a,b] = \{x: a < x \le b\}.$$

If $E \subset [-\infty, \infty]$ and $E \neq \varnothing$, the least upper bound (supremum) and greatest lower bound (infimum) of E exist in $[-\infty, \infty]$ and are denoted by $\sup E$ and $\inf E$.

Sometimes (but only when $\sup E \in E$) we write $\max E$ for $\sup E$.

The symbol

$$f: X \to Y$$

means that f is a *function* (or *mapping* or *transformation*) of the set X into the set Y; i.e., f assigns to each $x \in X$ an element $f(x) \in Y$. If $A \subset X$ and $B \subset Y$, the *image* of A and the *inverse image* (or pre-image) of B are

$$f(A) = \{y: y = f(x) \text{ for some } x \in A\},$$
$$f^{-1}(B) = \{x: f(x) \in B\}.$$

Note that $f^{-1}(B)$ may be empty although $B \neq \varnothing$.

The *domain* of f is X. The *range* of f is $f(X)$.

If $f(X) = Y$, f is said to map X *onto* Y.

We write $f^{-1}(y)$, instead of $f^{-1}(\{y\})$, for every $y \in Y$. If $f^{-1}(y)$ consists of at most one point, for each $y \in Y$, f is said to be *one-to-one*. If f is one-to-one, then f^{-1} is a function with domain $f(X)$ and range X.

If $f: X \to [-\infty, \infty]$ and $E \subset X$, it is customary to write $\sup_{x \in E} f(x)$ rather than $\sup f(E)$.

If $f: X \to Y$ and $g: Y \to Z$, the composite function $g \circ f: X \to Z$ is defined by the formula

$$(g \circ f)(x) = g(f(x)) \qquad (x \in X).$$

The Concept of Measurability

The class of measurable functions plays a fundamental role in integration theory. It has some basic properties in common with another most important class of functions, namely, the continuous ones. It is helpful to keep these similarities in mind. Our presentation is therefore organized in such a way that the analogies between the concepts *topological space, open set,* and *continuous function,* on the one hand, and *measurable space, measurable set,* and *measurable function,* on the other, are strongly emphasized. It seems that the relations between these concepts emerge most clearly when the setting is quite abstract, and this (rather than a desire for mere generality) motivates our approach to the subject.

1.2 Definition

(*a*) A collection τ of subsets of a set X is said to be a *topology in X* if τ has the following three properties:

(i) $\varnothing \in \tau$ and $X \in \tau$.
(ii) If $V_i \in \tau$ for $i = 1, \ldots, n$, then $V_1 \cap V_2 \cap \cdots \cap V_n \in \tau$.
(iii) If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, or uncountable), then $\bigcup_\alpha V_\alpha \in \tau$.

(*b*) If τ is a topology in X, then X is called a *topological space,* and the members of τ are called the *open sets* in X.

(*c*) If X and Y are topological spaces and if f is a mapping of X into Y, then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

1.3 Definition

(*a*) A collection $\mathfrak{M}$ of subsets of a set X is said to be a *σ-algebra in X* if $\mathfrak{M}$ has the following three properties:

(i) $X \in \mathfrak{M}$.
(ii) If $A \in \mathfrak{M}$, then $A^c \in \mathfrak{M}$, where A^c is the complement of A relative to X.
(iii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathfrak{M}$ for $n = 1, 2, 3, \ldots$, then $A \in \mathfrak{M}$.

(*b*) If $\mathfrak{M}$ is a σ-algebra in X, then X is called a *measurable space,* and the members of $\mathfrak{M}$ are called the *measurable sets* in X.

(*c*) If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

It would perhaps be more satisfactory to apply the term "measurable space" to the ordered pair $(X,\mathfrak{M})$, rather than to X. After all, X is a set, and X has not been changed in any way by the fact that we now also have a σ-algebra of its subsets in mind. Similarly, a topological space is an ordered pair (X,τ). But if this sort of thing were systematically done in all mathematics, the terminology would become awfully cumbersome. We shall discuss this again at somewhat greater length in Sec. 1.21.

1.4 Comments on Definition 1.2 The most familiar topological spaces are the *metric spaces.* We shall assume some familiarity with metric spaces but shall give the basic definitions, for the sake of completeness.

A *metric space* is a set X in which a *distance function* (or *metric*) ρ is defined, with the following properties:

(*a*) $0 \le \rho(x,y) < \infty$ for all x and $y \in X$.
(*b*) $\rho(x,y) = 0$ if and only if $x = y$.
(*c*) $\rho(x,y) = \rho(y,x)$ for all x and $y \in X$.
(*d*) $\rho(x,y) \le \rho(x,z) + \rho(z,y)$ for all x, y, and $z \in X$.

Property (*d*) is called the *triangle inequality.*

If $x \in X$ and $r \ge 0$, the *open ball* with center at x and radius r is the set $\{y \in X : \rho(x,y) < r\}$.

If X is a metric space and if τ is the collection of all sets $E \subset X$ which are arbitrary unions of open balls, then τ is a topology in X. This is not hard to verify; the intersection property depends on the fact that if $x \in B_1 \cap B_2$, where B_1 and B_2 are open balls, then x is the center of an open ball $B \subset B_1 \cap B_2$. We leave this as an exercise.

For instance, in the real line R^1 a set is open if and only if it is a union of open segments (a,b). In the plane R^2, the open sets are those which are unions of open circular discs.

Another topological space, which we shall encounter frequently, is the extended real line $[-\infty, \infty]$; its topology is defined by declaring the following sets to be open: (a,b), $[-\infty,a)$, $(a,\infty]$, and any union of segments of this type.

The definition of continuity given in Sec. 1.2(*c*) is a global one. Frequently it is desirable to define continuity locally: A mapping f of X into Y is said to be *continuous at the point* $x_0 \in X$ if to every neighborhood V of $f(x_0)$ there corresponds a neighborhood W of x_0 such that $f(W) \subset V$.

(A *neighborhood* of a point x is, by definition, an open set which contains x.)

For metric spaces, this local definition is of course the same as the usual epsilon-delta definition.

The following easy proposition relates the two definitions of continuity in the expected manner:

1.5 Proposition *Let X and Y be topological spaces. A mapping f of X into Y is continuous if and only if f is continuous at every point of X.*

PROOF If f is continuous and $x_0 \in X$, then $f^{-1}(V)$ is a neighborhood of x_0, for every neighborhood V of $f(x_0)$. Since $f(f^{-1}(V)) \subset V$, it follows that f is continuous at x_0.

If f is continuous at every point of X and if V is open in Y, every point $x \in f^{-1}(V)$ has a neighborhood W_x such that $f(W_x) \subset V$. Hence $W_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ is the union of the open sets W_x, so $f^{-1}(V)$ is itself open. Thus f is continuous.

1.6 Comments on Definition 1.3 Let $\mathfrak{M}$ be a σ-algebra in a set X. Referring to Properties (i) to (iii) of Definition 1.3(a), we immediately derive the following:

(a) Since $\varnothing = X^c$, (i) and (ii) imply that $\varnothing \in \mathfrak{M}$.
(b) Taking $A_{n+1} = A_{n+2} = \cdots = \varnothing$ in (iii), we see that $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathfrak{M}$ if $A_i \in \mathfrak{M}$ for $i = 1, \ldots, n$.
(c) Since

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c\right)^c,$$

$\mathfrak{M}$ is closed under the formation of countable (and also finite) intersections.
(d) Since $A - B = B^c \cap A$, we have $A - B \in \mathfrak{M}$ if $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$.

The prefix σ refers to the fact that (iii) is required to hold for all *countable* unions of members of $\mathfrak{M}$. If (iii) is required for finite unions only, then $\mathfrak{M}$ is called an *algebra* of sets.

1.7 Theorem *Let Y and Z be topological spaces, and let $g: Y \to Z$ be continuous.*

(a) *If X is a topological space, if $f: X \to Y$ is continuous, and if $h = g \circ f$, then $h: X \to Z$ is continuous.*
(b) *If X is a measurable space, if $f: X \to Y$ is measurable, and if $h = g \circ f$, then $h: X \to Z$ is measurable.*

Stated informally, continuous functions of continuous functions are continuous; continuous functions of measurable functions are measurable.

PROOF If V is open in Z, then $g^{-1}(V)$ is open in Y, and

$$h^{-1}(V) = f^{-1}(g^{-1}(V)).$$

If f is continuous, it follows that $h^{-1}(V)$ is open, proving (a).
If f is measurable, it follows that $h^{-1}(V)$ is measurable, proving (b).

1.8 Theorem *Let u and v be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define*

$$h(x) = \Phi(u(x),v(x))$$

for $x \in X$. Then $h: X \to Y$ is measurable.

PROOF Put $f(x) = (u(x),v(x))$. Then f maps X into the plane. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove the measurability of f.

If R is any open rectangle in the plane, with sides parallel to the axes, then R is the cartesian product of two segments I_1 and I_2, and

$$f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2),$$

which is measurable, by our assumption on u and v. Every open set V in the plane is a countable union of such rectangles R_i, and since

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{\infty} R_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(R_i),$$

$f^{-1}(V)$ is measurable.

1.9 Let X be a measurable space. The following propositions are corollaries of Theorems 1.7 and 1.8:

(a) *If $f = u + iv$, where u and v are real measurable functions on X, then f is a complex measurable function on X.*
This follows from Theorem 1.8, with $\Phi(z) = z$.
(b) *If $f = u + iv$ is a complex measurable function on X, then u, v, and $|f|$ are real measurable functions on X.*
This follows from Theorem 1.7, with $g(z) = \mathrm{Re}\,(z)$, $\mathrm{Im}\,(z)$, and $|z|$.
(c) *If f and g are complex measurable functions on X, then so are $f + g$ and fg.*
For real f and g this follows from Theorem 1.8, with

$$\Phi(s,t) = s + t$$

and $\Phi(s,t) = st$. The complex case then follows from (a) and (b).
(d) *If E is a measurable set in X and if*

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then χ_E is a measurable function.
This is obvious. We call χ_E the *characteristic function* of the set E. The letter χ will be reserved for characteristic functions throughout this book.

(e) *If f is a complex measurable function on X, there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.*

PROOF Let $E = \{x: f(x) = 0\}$, let Y be the complex plane with the origin removed, define $\varphi(z) = z/|z|$ for $z \in Y$, and put

$$\alpha(x) = \varphi(f(x) + \chi_E(x)) \qquad (x \in X).$$

If $x \in E$, $\alpha(x) = 1$; if $x \notin E$, $\alpha(x) = f(x)/|f(x)|$. Since φ is continuous on Y and since E is measurable (why?), the measurability of α follows from (c), (d), and Theorem 1.7.

We now show that σ-algebras exist in great profusion.

1.10 Theorem *If $\mathfrak{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathfrak{M}^*$ in X such that $\mathfrak{F} \subset \mathfrak{M}^*$.*

This $\mathfrak{M}^*$ is sometimes called the σ-algebra generated by $\mathfrak{F}$.

PROOF Let Ω be the family of all σ-algebras $\mathfrak{M}$ in X which contain $\mathfrak{F}$. Since the collection of all subsets of X is such a σ-algebra, Ω is not empty. Let $\mathfrak{M}^*$ be the intersection of all $\mathfrak{M} \in \Omega$. It is clear that $\mathfrak{F} \subset \mathfrak{M}^*$ and that $\mathfrak{M}^*$ lies in every σ-algebra in X which contains $\mathfrak{F}$. To complete the proof, we have to show that $\mathfrak{M}^*$ is itself a σ-algebra.

If $A_n \in \mathfrak{M}^*$ for $n = 1, 2, 3, \ldots$, and if $\mathfrak{M} \in \Omega$, then $A_n \in \mathfrak{M}$, so $\bigcup A_n \in \mathfrak{M}$, since $\mathfrak{M}$ is a σ-algebra. Since $\bigcup A_n \in \mathfrak{M}$ for *every* $\mathfrak{M} \in \Omega$, we conclude that $\bigcup A_n \in \mathfrak{M}^*$. The other two defining properties of a σ-algebra are verified in the same manner.

1.11 Borel Sets Let X be a topological space. By Theorem 1.10, there exists a smallest σ-algebra $\mathfrak{B}$ in X such that every open set in X belongs to $\mathfrak{B}$. The members of $\mathfrak{B}$ are called the *Borel sets* of X.

In particular, closed sets are Borel sets (being, by definition, the complements of open sets), and so are all countable unions of closed sets and all countable intersections of open sets. These last two are called F_σ's and G_δ's, respectively, and play a considerable role. The notation is due to Hausdorff. The letters F and G were used for closed and open sets, respectively, and σ refers to union (Summe), δ to intersection (Durchschnitt). For example, every half-open interval $[a,b)$ is a G_δ and an F_σ in R^1.

Since $\mathfrak{B}$ is a σ-algebra, we may now regard X as a measurable space, with the Borel sets playing the role of the measurable sets; more concisely, we consider the measurable space $(X,\mathfrak{B})$. If $f: X \to Y$ is a continuous mapping of X, where Y is any topological space, then it is evident from the definitions that $f^{-1}(V) \in \mathfrak{B}$ for every open set V in Y. In other words, *every continuous mapping of X is Borel measurable.*

If Y is the real line or the complex plane, the Borel measurable mappings will be called *Borel functions.*

1.12 Theorem *Suppose $\mathfrak{M}$ is a σ-algebra in X and Y is a topological space. Let f map X into Y.*

(a) *If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ-algebra in Y.*
(b) *If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathfrak{M}$.*
(c) *If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ for every real α, then f is measurable.*

PROOF (a) follows from the relations

$$f^{-1}(Y) = X, \qquad f^{-1}(Y - A) = X - f^{-1}(A),$$

and $$f^{-1}(A_1 \cup A_2 \cup \cdots) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots.$$

To prove (b), let Ω be as in (a); the measurability of f implies that Ω contains all open sets in Y, and since Ω is a σ-algebra, Ω contains all Borel sets in Y.

To prove (c), let Ω be the collection of all $E \subset [-\infty, \infty]$ such that $f^{-1}(E) \in \mathfrak{M}$. Since Ω is a σ-algebra in $[-\infty, \infty]$, and since $(\alpha, \infty] \in \Omega$ for all real α, the same is true of the sets

$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} \left[-\infty, \alpha - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} \left(\alpha - \frac{1}{n}, \infty\right]^c$$

and $$(\alpha,\beta) = [-\infty,\beta) \cap (\alpha, \infty].$$

Since every open set in $[-\infty, \infty]$ is a countable union of segments of the above types, Ω contains every open set, so f is measurable.

1.13 Definition Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$, and put

(1) $$b_k = \sup \{a_k, a_{k+1}, a_{k+2}, \ldots\} \qquad (k = 1, 2, 3, \ldots)$$

and

(2) $$\beta = \inf \{b_1, b_2, b_3, \ldots\}.$$

We call β the *upper limit* of $\{a_n\}$, and write

(3) $$\beta = \limsup_{n\to\infty} a_n.$$

The following properties are easily verified: First, $b_1 \geq b_2 \geq b_3 \geq \cdots$, so that $b_k \to \beta$ as $k \to \infty$; secondly, there is a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} \to \beta$ as $i \to \infty$, and β is the largest number with this property.

The *lower limit* is defined analogously: simply interchange sup and inf in (1) and (2). Note that

(4) $$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} (-a_n).$$

If $\{a_n\}$ converges, then evidently

(5) $$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n.$$

Suppose $\{f_n\}$ is a sequence of extended-real functions on a set X. Then $\sup_n f_n$ and $\limsup_{n\to\infty} f_n$ are the functions defined on X by

(6) $$(\sup_n f_n)(x) = \sup_n (f_n(x)),$$

(7) $$(\limsup_{n\to\infty} f_n)(x) = \limsup_{n\to\infty} (f_n(x)).$$

If

(8) $$f(x) = \lim_{n\to\infty} f_n(x),$$

the limit being assumed to exist at every $x \in X$, then we call f the *pointwise limit* of the sequence $\{f_n\}$.

1.14 Theorem *If $f_n: X \to [-\infty, \infty]$ is measurable, for $n = 1, 2, 3, \ldots$, and*

$$g = \sup_{n\geq 1} f_n, \qquad h = \limsup_{n\to\infty} f_n,$$

then g and h are measurable.

PROOF $g^{-1}((\alpha, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])$. Hence Theorem 1.12(*c*) implies that g is measurable. The same result holds of course with inf in place of sup, and since

$$h = \inf_{k\geq 1} \{\sup_{i\geq k} f_i\},$$

it follows that h is measurable.

Corollaries

(*a*) *The limit of every pointwise convergent sequence of complex measurable functions is measurable.*

(*b*) *If f and g are measurable (with range in $[-\infty, \infty]$), then so are* $\max\{f,g\}$ *and* $\min\{f,g\}$. *In particular, this is true of the functions*

$$f^+ = \max\{f,0\} \quad \text{and} \quad f^- = -\min\{f,0\}.$$

1.15 The above functions f^+ and f^- are called the *positive* and *negative parts* of f. We have $|f| = f^+ + f^-$ and $f = f^+ - f^-$, a standard representation of f as a difference of two nonnegative functions, with a certain minimum property:

Proposition *If $f = g - h$, $g \geq 0$, and $h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.*

PROOF $f \leq g$ and $0 \leq g$ clearly implies $\max\{f,0\} \leq g$.

Simple Functions

1.16 Definition A function s on a measurable space X whose range consists of only finitely many points in $[0, \infty)$ will be called a *simple function.*

(Sometimes it is convenient to call *any* function with finite range simple. The above situation is, however, the one we shall be mainly interested in. Note that we explicitly exclude ∞ from the values of a simple function.)

If $\alpha_1, \ldots, \alpha_n$ are the distinct values of a simple function s, and if $A_i = \{x: s(x) = \alpha_i\}$, then clearly

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where χ_{A_i} is the characteristic function of A_i, as defined in Sec. 1.9(*d*).

It is also clear that s is measurable if and only if each of the sets A_i is measurable.

1.17 Theorem *Let $f: X \to [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that*

(*a*) $0 \leq s_1 \leq s_2 \leq \cdots \leq f$.
(*b*) $s_n(x) \to f(x)$ *as* $n \to \infty$, *for every* $x \in X$.

PROOF For $n = 1, 2, 3, \ldots$, and for $1 \leq i \leq n2^n$, define

(1) $$E_{n,i} = f^{-1}\left(\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \quad \text{and} \quad F_n = f^{-1}([n, \infty])$$

and put

(2) $$s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} \chi_{E_{n,i}} + n\chi_{F_n}.$$

Theorem 1.12(*b*) shows that $E_{n,i}$ and F_n are measurable sets. It is easily seen that the functions (2) satisfy (*a*). If x is such that $f(x) < \infty$, then $s_n(x) \geq f(x) - 2^{-n}$ as soon as n is large enough; if $f(x) = \infty$, then $s_n(x) = n$; this proves (*b*).

It should be observed that the preceding construction yields a uniformly convergent sequence $\{s_n\}$ if f is bounded.

Elementary Properties of Measures

1.18 Definition

(a) A *positive measure* is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is *countably additive*. This means that if $\{A_i\}$ is a *disjoint* countable collection of members of $\mathfrak{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{1}$$

To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.

(b) A *measure space* is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

(c) A *complex measure* is a complex-valued countably additive function defined on a σ-algebra.

Note: What we have called a *positive measure* is frequently just called a *measure;* we add the word "positive" for emphasis. If $\mu(E) = 0$ for every $E \in \mathfrak{M}$, then μ is a positive measure, by our definition. The value ∞ is admissible for a positive measure; but when we talk of a complex measure μ, it is understood that $\mu(E)$ is a complex number, for every $E \in \mathfrak{M}$. The *real measures* form a subclass of the complex ones, of course.

1.19 Theorem *Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then*

(a) $\mu(\varnothing) = 0$.

(b) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ *if* $A_1, \ldots, A_n$ *are pairwise disjoint members of* $\mathfrak{M}$.

(c) $A \subset B$ *implies* $\mu(A) \le \mu(B)$ *if* $A \in \mathfrak{M}$, $B \in \mathfrak{M}$.

(d) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$, *and*

$$A_1 \subset A_2 \subset A_3 \subset \cdots.$$

(e) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$,

$$A_1 \supset A_2 \supset A_3 \supset \cdots,$$

and $\mu(A_1)$ *is finite.*

As the proof will show, these properties, with the exception of (c), also hold for complex measures; (b) is called *finite additivity;* (c) is called *monotonicity*.

PROOF

(a) Take $A \in \mathfrak{M}$ so that $\mu(A) < \infty$, and take $A_1 = A$ and $A_2 = A_3 = \cdots = \varnothing$ in 1.18(1).

(b) Take $A_{n+1} = A_{n+2} = \cdots = \varnothing$ in 1.18(1).

(c) Since $B = A \cup (B - A)$ and $A \cap (B - A) = \varnothing$, (b) gives $\mu(B) = \mu(A) + \mu(B - A) \ge \mu(A)$.

(d) Put $B_1 = A_1$, $B_n = A_n - A_{n-1}$ for $n = 2, 3, 4, \ldots$. Then $B_n \in \mathfrak{M}$, $B_i \cap B_j = \varnothing$ if $i \ne j$, $A_n = B_1 \cup \cdots \cup B_n$, and $A = \bigcup_{i=1}^{\infty} B_i$. Hence

$$\mu(A_n) = \sum_{i=1}^{n} \mu(B_i) \qquad \text{and} \qquad \mu(A) = \sum_{i=1}^{\infty} \mu(B_i).$$

Now (d) follows, by the definition of the sum of an infinite series.

(e) Put $C_n = A_1 - A_n$. Then $C_1 \subset C_2 \subset C_3 \subset \cdots$,

$$\mu(C_n) = \mu(A_1) - \mu(A_n),$$

$A_1 - A = \bigcup C_n$, and so (d) shows that

$$\mu(A_1) - \mu(A) = \mu(A_1 - A) = \lim_{n\to\infty} \mu(C_n) = \mu(A_1) - \lim_{n\to\infty} \mu(A_n).$$

This implies (e).

1.20 Examples The construction of interesting measure spaces requires some labor, as we shall see. However, a few simple-minded examples can be given immediately:

(a) For any $E \subset X$, where X is any set, define $\mu(E) = \infty$ if E is an infinite set, and let $\mu(E)$ be the number of points in E if E is finite. This μ is called the *counting measure* on X.

(b) Fix $x_0 \in X$, define $\mu(E) = 1$ if $x_0 \in E$ and $\mu(E) = 0$ if $x_0 \notin E$, for any $E \subset X$. This μ may be called the *unit mass* concentrated at x_0.

(c) Let μ be the counting measure on the set $\{1,2,3, \ldots\}$, let $A_n = \{n, n+1, n+2, \ldots\}$. Then $\bigcap A_n = \varnothing$ but $\mu(A_n) = \infty$ for $n = 1, 2, 3, \ldots$. This shows that the hypothesis

$$\text{“}\mu(A_1) < \infty\text{”}$$

is not superfluous in Theorem 1.19(e).

1.21 A Comment on Terminology One frequently sees measure spaces referred to as "ordered triples" $(X,\mathfrak{M},\mu)$ where X is a set, $\mathfrak{M}$ is a σ-algebra in X, and μ is a measure defined on $\mathfrak{M}$. Similarly, measurable spaces are "ordered pairs" $(X,\mathfrak{M})$. This is logically all right, and often convenient, though somewhat redundant. For instance, in $(X,\mathfrak{M})$ the set X is merely the largest member of $\mathfrak{M}$, so if we know $\mathfrak{M}$ we also know X. Similarly, every measure has a σ-algebra for its domain, by definition, so if we know a measure μ we also know the σ-algebra $\mathfrak{M}$ on which μ is defined and we know the set X in which $\mathfrak{M}$ is a σ-algebra.

It is therefore perfectly legitimate to use expressions like "Let μ be a measure" or, if we wish to emphasize the σ-algebra or the set in question, to say "Let μ be a measure on $\mathfrak{M}$" or "Let μ be a measure on X."

What is logically rather meaningless but customary (and we shall often follow mathematical custom rather than logic) is to say "Let X be a measure space"; the emphasis should not be on the set, but on the measure. Of course, when this wording is used, it is tacitly understood that there is a measure defined on some σ-algebra in X and that it is this measure which is really under discussion.

Similarly, a topological space is an ordered pair (X,τ), where τ is a topology in the set X, and the significant data are contained in τ, not in X, but "the topological space X" is what one talks about.

This sort of tacit convention is used throughout mathematics. Most mathematical systems are sets with some class of distinguished subsets or some binary operations or some relations (which are required to have certain properties), and one can list these and then describe the system as an ordered pair, triple, etc., depending on what is needed. For instance, the real line may be described as a quadruple $(R^1,+,\cdot,<)$, where $+$, $\cdot$, and $<$ satisfy the axioms of a complete archimedean ordered field. But it is a safe bet that very few mathematicians think of the real field as an ordered quadruple.

Arithmetic in $[0,\infty]$

1.22 Throughout integration theory, one inevitably encounters ∞. One reason is that one wants to be able to integrate over sets of infinite measure; after all, the real line has infinite length. Another reason is that even if one is primarily interested in real-valued functions, the lim sup of a sequence of positive real functions or the sum of a sequence of positive real functions may well be ∞ at some points, and much of the elegance of theorems like 1.26 and 1.27 would be lost if one had to make some special provisions whenever this occurs.

Let us define $a + \infty = \infty + a = \infty$ if $0 \le a \le \infty$, and

$$a \cdot \infty = \infty \cdot a = \begin{cases} \infty & \text{if } 0 < a \le \infty \\ 0 & \text{if } a = 0; \end{cases}$$

It may seem strange to define $0 \cdot \infty = 0$. However, one verifies without difficulty that with this definition the *commutative, associative, and distributive laws hold in* $[0,\infty]$ *without any restriction.*

The cancellation laws have to be treated with some care: $a + b = a + c$ implies $b = c$ only when $a < \infty$, and $ab = ac$ implies $b = c$ only when $0 < a < \infty$.

Observe that the following useful proposition holds:

If $0 \le a_1 \le a_2 \le \cdots$, $0 \le b_1 \le b_2 \le \cdots$, $a_n \to a$, *and* $b_n \to b$, *then* $a_nb_n \to ab$.

If we combine this with Theorems 1.17 and 1.14, we see that *sums and products of measurable functions into* $[0,\infty]$ *are measurable.*

Integration of Positive Functions

In this section, $\mathfrak{M}$ will be a σ-algebra in a set X and μ will be a positive measure on $\mathfrak{M}$.

1.23 Definition If s is a measurable simple function on X, of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \tag{1}$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct values of s (compare Definition 1.16), and if $E \in \mathfrak{M}$, we define

$$\int_E s\, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E). \tag{2}$$

The convention $0 \cdot \infty = 0$ is used here; it may happen that $\alpha_i = 0$ for some i and that $\mu(A_i \cap E) = \infty$.

If $f\colon X \to [0,\infty]$ is measurable, and $E \in \mathfrak{M}$, we define

$$\int_E f\, d\mu = \sup \int_E s\, d\mu, \tag{3}$$

the supremum being taken over all simple measurable functions s such that $0 \le s \le f$.

The left member of (3) is called the *Lebesgue integral* of f over E, with respect to the measure μ. It is a number in $[0,\infty]$.

Observe that we apparently have two definitions for $\int_E f\, d\mu$ if f is simple, namely, (2) and (3). However, these assign the same value to the integral, since f is, in this case, the largest of the functions s which occur on the right of (3).

1.24 The following propositions are immediate consequences of the definitions. The functions and sets occurring in them are assumed to be measurable:

(a) *If* $0 \le f \le g$, *then* $\int_E f\,d\mu \le \int_E g\,d\mu$.
(b) *If* $A \subset B$ *and* $f \ge 0$, *then* $\int_A f\,d\mu \le \int_B f\,d\mu$.
(c) *If* $f \ge 0$ *and* c *is a constant*, $0 \le c < \infty$, *then*

$$\int_E cf\,d\mu = c\int_E f\,d\mu.$$

(d) *If* $f(x) = 0$ *for all* $x \in E$, *then* $\int_E f\,d\mu = 0$, *even if* $\mu(E) = \infty$.
(e) *If* $\mu(E) = 0$, *then* $\int_E f\,d\mu = 0$, *even if* $f(x) = \infty$ *for every* $x \in E$.
(f) *If* $f \ge 0$, *then* $\int_E f\,d\mu = \int_X \chi_E f\,d\mu$.

This last result shows that we could have restricted our definition of integration to integrals over all of X, without losing any generality. If we wanted to integrate over subsets, we could then use (*f*) as the definition. It is purely a matter of taste which definition is preferred.

One may also remark here that every measurable subset E of a measure space X is again a measure space, in a perfectly natural way: The new measurable sets are simply those measurable subsets of X which lie in E, and the measure is unchanged, except that its domain is restricted. This shows again that as soon as we have integration defined over every measure space, we automatically have it defined over every measurable subset of every measure space.

1.25 Proposition *Let* s *and* t *be measurable simple functions on* X. *For* $E \in \mathfrak{M}$, *define*

(1) $$\varphi(E) = \int_E s\,d\mu.$$

Then φ *is a measure on* $\mathfrak{M}$. *Also*

(2) $$\int_X (s+t)\,d\mu = \int_X s\,d\mu + \int_X t\,d\mu.$$

(This proposition contains provisional forms of Theorems 1.27 and 1.29.)

PROOF If s is as in Definition 1.23, and if $E_1, E_2, \ldots$ are disjoint members of $\mathfrak{M}$ whose union is E, the countable additivity of μ shows that

$$\varphi(E) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) = \sum_{i=1}^{n} \alpha_i \sum_{r=1}^{\infty} \mu(A_i \cap E_r)$$
$$= \sum_{r=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E_r) = \sum_{r=1}^{\infty} \varphi(E_r).$$

Also, $\varphi(\varnothing) = 0$, so that φ is not identically ∞.

Next, let s be as before, let $\beta_1, \ldots, \beta_m$ be the distinct values of t, and let $B_j = \{x : t(x) = \beta_j\}$. If $E_{ij} = A_i \cap B_j$, then

$$\int_{E_{ij}} (s+t)\,d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$$

and $$\int_{E_{ij}} s\,d\mu + \int_{E_{ij}} t\,d\mu = \alpha_i\mu(E_{ij}) + \beta_j\mu(E_{ij}).$$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1 \le i \le n,\ 1 \le j \le m)$, the first half of our proposition implies that (2) holds.

We now come to the interesting part of the theory. One of its most remarkable features is the ease with which it handles limit operations.

1.26 Lebesgue's Monotone Convergence Theorem *Let* $\{f_n\}$ *be a sequence of measurable functions on* X *and suppose that*

(a) $0 \le f_1(x) \le f_2(x) \le \cdots \le \infty$ *for every* $x \in X$,
(b) $f_n(x) \to f(x)$ *as* $n \to \infty$, *for every* $x \in X$.

Then f *is measurable, and*

$$\int_X f_n\,d\mu \to \int_X f\,d\mu \qquad \text{as } n \to \infty.$$

PROOF Since $\int f_n \le \int f_{n+1}$, there exists an $\alpha \in [0,\infty]$ such that

(1) $$\int_X f_n\,d\mu \to \alpha \qquad \text{as } n \to \infty.$$

By Theorem 1.14, f is measurable. Since $f_n \le f$, we have $\int f_n \le \int f$ for every n, so (1) implies

(2) $$\alpha \le \int_X f\,d\mu.$$

Let s be any simple measurable function such that $0 \le s \le f$, let c be a constant, $0 < c < 1$, and define

(3) $$E_n = \{x : f_n(x) \ge cs(x)\} \qquad (n = 1, 2, 3, \ldots).$$

Each E_n is measurable, $E_1 \subset E_2 \subset E_3 \subset \cdots$, and $X = \bigcup E_n$. For if $f(x) = 0$, then $x \in E_1$; and if $f(x) > 0$, then $cs(x) < f(x)$, since $c < 1$; hence $x \in E_n$ for some n. Also

(4) $$\int_X f_n\,d\mu \ge \int_{E_n} f_n\,d\mu \ge c\int_{E_n} s\,d\mu \qquad (n = 1, 2, 3, \ldots).$$

Let $n \to \infty$, applying Proposition 1.25 and Theorem 1.19(*d*) to the last integral in (4). The result is

(5) $$\alpha \ge c\int_X s\,d\mu.$$

Since (5) holds for every $c < 1$, we have

$$\alpha \geq \int_X s\,d\mu \tag{6}$$

for every simple measurable s satisfying $0 \leq s \leq f$, so that

$$\alpha \geq \int_X f\,d\mu. \tag{7}$$

The theorem follows from (1), (2), and (7).

1.27 Theorem *If $f_n: X \to [0, \infty]$ is measurable, for $n = 1, 2, 3, \ldots$, and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in X), \tag{1}$$

then

$$\int_X f\,d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu. \tag{2}$$

PROOF First, there are sequences $\{s_i'\}$, $\{s_i''\}$ of simple measurable functions such that $s_i' \to f_1$ and $s_i'' \to f_2$, as in Theorem 1.17. If $s_i = s_i' + s_i''$, then $s_i \to f_1 + f_2$, and the monotone convergence theorem, combined with Proposition 1.25, shows that

$$\int_X (f_1 + f_2)\,d\mu = \int_X f_1\,d\mu + \int_X f_2\,d\mu. \tag{3}$$

Next, put $g_N = f_1 + \cdots + f_N$. The sequence $\{g_N\}$ converges monotonically to f, and if we apply induction to (3) we see that

$$\int_X g_N\,d\mu = \sum_{n=1}^{N} \int_X f_n\,d\mu. \tag{4}$$

Applying the monotone convergence theorem once more, we obtain (2), and the proof is complete.

If we let μ be the counting measure on a countable set, Theorem 1.27 is a statement about double series of nonnegative real numbers (which can of course be proved by elementary means):

Corollary *If $a_{ij} \geq 0$ for i and $j = 1, 2, 3, \ldots$, then*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

1.28 Fatou's Lemma *If $f_n: X \to [0, \infty]$ is measurable, for each positive integer n, then*

$$\int_X \left(\liminf_{n\to\infty} f_n\right) d\mu \leq \liminf_{n\to\infty} \int_X f_n\,d\mu. \tag{1}$$

Strict inequality can occur in (1); see Exercise 2.

PROOF Put

$$g_k(x) = \inf_{i \geq k} f_i(x) \qquad (k = 1, 2, 3, \ldots;\ x \in X). \tag{2}$$

Then $g_k \leq f_k$, so that

$$\int_X g_k\,d\mu \leq \int_X f_k\,d\mu \qquad (k = 1, 2, 3, \ldots). \tag{3}$$

Also, $0 \leq g_1 \leq g_2 \leq \cdots$, and g_k is measurable, by Theorem 1.14, and $g_k(x) \to \liminf f_n(x)$ as $k \to \infty$, by Definition 1.13. The monotone convergence theorem therefore shows that the left side of (3) tends to the left side of (1), as $k \to \infty$. Hence (1) follows from (3).

1.29 Theorem *Suppose $f: X \to [0, \infty]$ is measurable, and*

$$\varphi(E) = \int_E f\,d\mu \qquad (E \in \mathfrak{M}). \tag{1}$$

Then φ is a measure on $\mathfrak{M}$, and

$$\int_X g\,d\varphi = \int_X gf\,d\mu \tag{2}$$

for every measurable g on X with range in $[0, \infty]$.

PROOF Let $E_1, E_2, E_3, \ldots$ be disjoint members of $\mathfrak{M}$ whose union is E. Observe that

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f \tag{3}$$

and that

$$\varphi(E) = \int_X \chi_E f\,d\mu, \qquad \varphi(E_j) = \int_X \chi_{E_j} f\,d\mu. \tag{4}$$

It now follows from Theorem 1.27 that

$$\varphi(E) = \sum_{j=1}^{\infty} \varphi(E_j). \tag{5}$$

Since $\varphi(\varnothing) = 0$, (5) proves that φ is a measure.

Next, (1) shows that (2) holds whenever $g = \chi_E$ for some $E \in \mathfrak{M}$. Hence (2) holds for every simple measurable function g, and the general case follows from the monotone convergence theorem.

Remark The second assertion of Theorem 1.29 is sometimes written in the form

$$d\varphi = f\,d\mu. \tag{6}$$

We assign no independent meaning to the symbols $d\varphi$ and $d\mu$; (6) merely means that (2) holds for every measurable $g \geq 0$.

Theorem 1.29 has a very important converse, the Radon-Nikodym theorem, which will be proved in Chap. 6.

Integration of Complex Functions

As before, μ will in this section be a positive measure on an arbitrary measurable space X.

1.30 Definition We define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which

$$\int_X |f|\, d\mu < \infty.$$

Note that the measurability of f implies that of $|f|$, as we saw in Proposition 1.9(*b*); hence the above integral is defined.

The members of $L^1(\mu)$ are called *Lebesgue integrable* functions (with respect to μ) or *summable functions*. The significance of the exponent 1 will become clear in Chap. 3.

1.31 Definition If $f = u + iv$, where u and v are real measurable functions on X, and if $f \in L^1(\mu)$, we define

$$(1) \qquad \int_E f\, d\mu = \int_E u^+\, d\mu - \int_E u^-\, d\mu + i \int_E v^+\, d\mu - i \int_E v^-\, d\mu$$

for every measurable set E.

Here u^+ and u^- are the positive and negative parts of u, as defined in Sec. 1.15; v^+ and v^- are similarly obtained from v. These four functions are measurable, real, and nonnegative; hence the four integrals on the right of (1) exist, by Definition 1.23. Furthermore, we have $u^+ \leq |u| \leq |f|$, etc., so that each of these four integrals is finite. Thus (1) defines the integral on the left as a complex number.

Occasionally it is desirable to define the integral of a measurable function f with range in $[-\infty, \infty]$ to be

$$(2) \qquad \int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu,$$

provided that at least one of the integrals on the right of (2) is finite. The left side of (2) is then a number in $[-\infty, \infty]$.

1.32 Theorem *Suppose f and $g \in L^1(\mu)$ and α and β are complex numbers. Then $\alpha f + \beta g \in L^1(\mu)$, and*

$$(1) \qquad \int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu.$$

PROOF The measurability of $\alpha f + \beta g$ follows from Proposition 1.9(*c*). By Sec. 1.24 and Theorem 1.27,

$$\int_X |\alpha f + \beta g|\, d\mu \leq \int_X (|\alpha|\,|f| + |\beta|\,|g|)\, d\mu = |\alpha| \int_X |f|\, d\mu + |\beta| \int_X |g|\, d\mu < \infty.$$

Thus $\alpha f + \beta g \in L^1(\mu)$.

To prove (1), it is clearly sufficient to prove

$$(2) \qquad \int_X (f + g)\, d\mu = \int_X f\, d\mu + \int_X g\, d\mu$$

and

$$(3) \qquad \int_X (\alpha f)\, d\mu = \alpha \int_X f\, d\mu,$$

and the general case of (2) will follow if we prove (2) for real f and g in $L^1(\mu)$.

Assuming this, and setting $h = f + g$, we have

$$h^+ - h^- = f^+ - f^- + g^+ - g^-$$

or

$$(4) \qquad h^+ + f^- + g^- = f^+ + g^+ + h^-.$$

By Theorem 1.27,

$$(5) \qquad \int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^-,$$

and since each of these integrals is finite, we may transpose and obtain (2).

That (3) holds if $\alpha \geq 0$ follows from Proposition 1.24(*c*). It is easy to verify that (3) holds if $\alpha = -1$, using relations like $(-u)^+ = u^-$. The case $\alpha = i$ is also easy: If $f = u + iv$, then

$$\int (if) = \int (iu - v) = \int (-v) + i \int u = -\int v + i \int u = i\left(\int u + i \int v\right) = i \int f.$$

Combining these cases with (2), we obtain (3) for any complex α.

1.33 Theorem *If $f \in L^1(\mu)$, then*

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu.$$

PROOF Put $z = \int_X f\, d\mu$. Since z is a complex number, there is a complex number α, with $|\alpha| = 1$, such that $\alpha z = |z|$. Let u be the real part of αf. Then $u \leq |\alpha f| = |f|$. Hence

$$\left| \int_X f\, d\mu \right| = \alpha \int_X f\, d\mu = \int_X \alpha f\, d\mu = \int_X u\, d\mu \leq \int_X |f|\, d\mu.$$

The third of the above equalities holds since the preceding ones show that $\int \alpha f\, d\mu$ is real.

We conclude this section with another important convergence theorem.

1.34 Lebesgue's Dominated Convergence Theorem *Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that*

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{1}$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \le g(x) \qquad (n = 1, 2, 3, \ldots ; x \in X), \tag{2}$$

then $f \in L^1(\mu)$,

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0, \tag{3}$$

and

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X f\, d\mu. \tag{4}$$

PROOF Since $|f| \le g$ and f is measurable, $f \in L^1(\mu)$. Since $|f_n - f| \le 2g$, Fatou's lemma applies to the functions $2g - |f_n - f|$ and yields

$$\int_X 2g\, d\mu \le \liminf_{n\to\infty} \int_X (2g - |f_n - f|)\, d\mu$$

$$= \int_X 2g\, d\mu + \liminf_{n\to\infty} \left(- \int_X |f_n - f|\, d\mu \right)$$

$$= \int_X 2g\, d\mu - \limsup_{n\to\infty} \int_X |f_n - f|\, d\mu.$$

Since $\int 2g\, d\mu$ is finite, we may subtract it and obtain

$$\limsup_{n\to\infty} \int_X |f_n - f|\, d\mu \le 0. \tag{5}$$

If a sequence of nonnegative real numbers fails to converge to 0, then its upper limit is positive. Thus (5) implies (3). By Theorem 1.33, applied to $f_n - f$, (3) implies (4).

The Role Played by Sets of Measure Zero

1.35 Definition Let P be a property which a point x may or may not have. For instance, P might be the property "$f(x) > 0$" if f is a given function, or it might be "$\{f_n(x)\}$ converges" if $\{f_n\}$ is a given sequence of functions.

If μ is a measure on a σ-algebra $\mathfrak{M}$ and if $E \in \mathfrak{M}$, the statement "P holds almost everywhere on E" (abbreviated to "P holds a.e. on E") means that there exists an $N \in \mathfrak{M}$ such that $\mu(N) = 0$, $N \subset E$, and P holds at every point of $E - N$. This concept of a.e. depends of course very strongly on the given measure, and we shall write "a.e. $[\mu]$" whenever clarity requires that the measure be indicated.

For example, if f and g are measurable functions and if

$$\mu(\{x : f(x) \ne g(x)\}) = 0, \tag{1}$$

we say that $f = g$ a.e. $[\mu]$ on X, and we may write $f \sim g$. This is easily seen to be an equivalence relation. The transitivity ($f \sim g$ and $g \sim h$ implies $f \sim h$) is a consequence of the fact that the union of two sets of measure 0 has measure 0.

Note that if $f \sim g$, then, for every $E \in \mathfrak{M}$,

$$\int_E f\, d\mu = \int_E g\, d\mu. \tag{2}$$

To see this, let N be the set which appears in (1); then E is the union of the disjoint sets $E - N$ and $E \cap N$; on $E - N$, $f = g$, and $\mu(E \cap N) = 0$.

Thus, generally speaking, sets of measure 0 are negligible in integration. It ought to be true that every subset of a negligible set is negligible. But it may happen that some set $N \in \mathfrak{M}$ with $\mu(N) = 0$ has a subset E which is not a member of $\mathfrak{M}$. Of course we can *define* $\mu(E) = 0$ in this case. But will this extension of μ still be a measure, i.e., will it still be defined on a σ-algebra? It is a pleasant fact that the answer is affirmative:

1.36 Theorem *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $\mathfrak{M}^*$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathfrak{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$ in this situation. Then $\mathfrak{M}^*$ is a σ-algebra, and μ is a measure on $\mathfrak{M}^*$.*

This extended measure μ is called *complete* since all subsets of sets of measure 0 are now measurable; the σ-algebra $\mathfrak{M}^*$ is called the *μ-completion* of $\mathfrak{M}$. The theorem says that every measure can be completed, so, whenever it is convenient, we may assume that any given measure is complete; this just gives us more measurable sets, hence more measurable functions. Most measures that one meets in the ordinary course of events are already complete, but there are exceptions; one of these will occur in the proof of Fubini's theorem in Chap. 7.

PROOF We verify the three defining properties of a σ-algebra. (i) $X \in \mathfrak{M}$, hence $X \in \mathfrak{M}^*$. (ii) If $A \subset E \subset B$, then $B^c \subset E^c \subset A^c$, and $A^c - B^c = B - A$. (iii) If $A_i \subset E_i \subset B_i$, $A = \bigcup A_i$, $E = \bigcup E_i$, and $B = \bigcup B_i$, then $A \subset E \subset B$ and

$$B - A \subset \bigcup_1^\infty (B_i - A_i),$$

so that $\mu(B - A) = 0$ if $\mu(B_i - A_i) = 0$ for $i = 1, 2, 3, \ldots$.

Next, we check that μ is well defined on $\mathfrak{M}^*$. Suppose $A \subset E \subset B$, $A_1 \subset E \subset B_1$, and $\mu(B - A) = \mu(B_1 - A_1) = 0$. Then

$$A - A_1 \subset B_1 - A_1,$$

so $\mu(A - A_1) = 0$. Similarly, $\mu(A_1 - A) = 0$. Hence

$$\mu(A) = \mu(A_1 \cap A) = \mu(A_1).$$

The countable additivity of μ on $\mathfrak{M}^*$ is obvious.

1.37 The fact that functions which are equal a.e. are indistinguishable as far as integration is concerned suggests that our definition of measurable function might profitably be enlarged. Let us call a function f defined on a set $E \in \mathfrak{M}$ *measurable on* X if $\mu(E^c) = 0$ and if $f^{-1}(V) \cap E$ is measurable for every open set V. If we define $f(x) = 0$ for $x \in E^c$, we obtain a measurable function on X, in the old sense. If our measure happens to be complete, we can define f on E^c in a perfectly arbitrary manner, and we still get a measurable function. The integral of f over any set $A \in \mathfrak{M}$ is independent of the definition of f on E^c; therefore this definition need not even be specified at all.

There are many situations where this occurs naturally. For instance, a function f on the real line may be differentiable only almost everywhere (with respect to Lebesgue measure), but under certain conditions it is still true that f is the integral of its derivative; this will be discussed in Chap. 8. Or a sequence $\{f_n\}$ of measurable functions on X may converge only almost everywhere; with our new definition of measurability, the limit is still a measurable function on X, and we do not have to cut down to the set on which convergence actually occurs.

To illustrate, let us state a corollary of Lebesgue's dominated convergence theorem in a form in which exceptional sets of measure zero are admitted:

1.38 Theorem *Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on X such that*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty. \tag{1}$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \tag{2}$$

converges for almost all x, $f \in L^1(\mu)$, and

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu. \tag{3}$$

PROOF Let S_n be the set on which f_n is defined, so that $\mu(S_n^c) = 0$. Put $\varphi(x) = \Sigma |f_n(x)|$, for $x \in S = \bigcap S_n$. Then $\mu(S^c) = 0$. By (1) and Theorem 1.27,

$$\int_S \varphi \, d\mu < \infty. \tag{4}$$

If $E = \{x \in S: \varphi(X) < \infty\}$, it follows from (4) that $\mu(E^c) = 0$. The series (2) converges absolutely for every $x \in E$, and if $f(x)$ is defined by (2) for $x \in E$, then $|f(x)| \le \varphi(x)$ on E, so that $f \in L^1(\mu)$ on E, by (4). If $g_n = f_1 + \cdots + f_n$, then $|g_n| \le \varphi$, $g_n(x) \to f(x)$ for all $x \in E$, and Theorem 1.34 gives (3) with E in place of X. This is equivalent to (3), since $\mu(E^c) = 0$.

Note that even if the f_n were defined at *every* point of X, (1) would only imply that (2) converges *almost everywhere*. Here are some other situations in which we can draw conclusions only almost everywhere:

1.39 Theorem

(*a*) *Suppose $f: X \to [0, \infty]$ is measurable, $E \in \mathfrak{M}$, and $\int_E f \, d\mu = 0$. Then $f = 0$ a.e. on E.*

(*b*) *Suppose $f \in L^1(\mu)$ and $\int_E f \, d\mu = 0$ for every $E \in \mathfrak{M}$. Then $f = 0$ a.e. on X.*

(*c*) *Suppose $f \in L^1(\mu)$ and*

$$\left| \int_X f \, d\mu \right| = \int_X |f| \, d\mu.$$

Then there is a constant α such that $\alpha f = |f|$ a.e. on X.

Note that (*c*) describes the condition under which equality holds in Theorem 1.33.

PROOF

(*a*) If $A_n = \{x \in E: f(x) > 1/n\}$, $n = 1, 2, 3, \ldots$, then

$$\frac{1}{n} \mu(A_n) \le \int_{A_n} f \, d\mu \le \int_E f \, d\mu = 0,$$

so that $\mu(A_n) = 0$. Since $\{x \in E: f(x) > 0\} = \bigcup A_n$, (*a*) follows.

(*b*) Put $f = u + iv$, let $E = \{x: u(x) \geq 0\}$. The real part of $\int_E f\,d\mu$ is then $\int_E u^+\,d\mu$. Hence $\int_E u^+\,d\mu = 0$, and (*a*) implies that $u^+ = 0$ a.e. We conclude similarly that

$$u^- = v^+ = v^- = 0 \text{ a.e.}$$

(*c*) Examine the proof of Theorem 1.33. Our present assumption implies that the last inequality in the proof of Theorem 1.33 must actually be an equality. Hence $\int(|f| - u)\,d\mu = 0$. Since $|f| - u \geq 0$, (*a*) shows that $|f| = u$ a.e. This says that the real part of αf is equal to $|\alpha f|$ a.e., hence $\alpha f = |\alpha f| = |f|$ a.e., which is the desired conclusion.

1.40 Theorem *Suppose $\mu(X) < \infty$, $f \in L^1(\mu)$, S is a closed set in the complex plane, and the averages*

$$A_E(f) = \frac{1}{\mu(E)} \int_E f\,d\mu$$

lie in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.

PROOF Let Δ be a closed circular disc (with center at α and radius $r > 0$, say) in the complement of S. Since S^c is the union of countably many such discs, it is enough to prove that $\mu(E) = 0$, where $E = f^{-1}(\Delta)$.

If we had $\mu(E) > 0$, then

$$|A_E(f) - \alpha| = \frac{1}{\mu(E)} \left| \int_E (f - \alpha)\,d\mu \right| \leq \frac{1}{\mu(E)} \int_E |f - \alpha|\,d\mu \leq r,$$

which is impossible, since $A_E(f) \in S$. Hence $\mu(E) = 0$.

1.41 Theorem *Let $\{E_k\}$ be a sequence of measurable sets in X, such that*

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty. \tag{1}$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.

PROOF If A is the set of all x which lie in infinitely many E_k, we have to prove that $\mu(A) = 0$. Put

$$g(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x) \qquad (x \in X). \tag{2}$$

For each x, each term in this series is either 0 or 1. Hence $x \in A$ if and only if $g(x) = \infty$. By Theorem 1.27, the integral of g over X is equal to the sum in (1). Thus $g \in L^1(\mu)$ and so $g(x) < \infty$ a.e.

Exercises

1 Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(*a*) $$\limsup_{n\to\infty} (-a_n) = -\liminf_{n\to\infty} a_n.$$

(*b*) $$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

provided none of the sums is of the form $\infty - \infty$.

(*c*) If $a_n \leq b_n$ for all n, then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

Show by an example that strict inequality can hold in (*b*).

2 Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

3 Suppose $f_n: X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu$$

and show that this conclusion does *not* follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

4 Prove that if f is a real function on a measurable space X such that $\{x: f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

5 Prove that the set of points at which a sequence of measurable real functions converges is a measurable set.

6 Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$.

7 Does there exist an infinite σ-algebra which has only countably many members?

8 Prove an analogue of Theorem 1.8 for n functions.

9 Prove the conclusion of Theorem 1.7(*b*) under the weaker hypothesis that g is Borel measurable; i.e., prove that Borel measurable functions of measurable functions are measurable.

10 Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu,$$

and show that the hypothesis "$\mu(X) < \infty$" cannot be omitted.

11 Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.

12 Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

13 Show that proposition 1.24(c) is also true for $c = \infty$.

2

Positive Borel Measures

Vector Spaces

2.1 Definition A *complex vector space* (or a vector space over the complex field) is a set V, whose elements are called *vectors* and in which two operations, called *addition* and *scalar multiplication*, are defined, with the following familiar algebraic properties:

To every pair of vectors x and y there corresponds a vector $x + y$, in such a way that $x + y = y + x$ and $x + (y + z) = (x + y) + z$; V contains a unique vector 0 (the *zero vector* or *origin* of V) such that $x + 0 = x$ for every $x \in V$; and to each $x \in V$ there corresponds a unique vector $-x$ such that $x + (-x) = 0$.

To each pair (α, x), where $x \in V$ and α is a scalar (in this context, the word *scalar* means *complex number*), there is associated a vector $\alpha x \in V$, in such a way that $1x = x$, $\alpha(\beta x) = (\alpha\beta)x$, and such that the two distributive laws

$$(1) \qquad \alpha(x + y) = \alpha x + \alpha y, \qquad (\alpha + \beta)x = \alpha x + \beta x$$

hold.

A *linear transformation* of a vector space V into a vector space V_1 is a mapping Λ of V into V_1 such that

$$(2) \qquad \Lambda(\alpha x + \beta y) = \alpha \Lambda x + \beta \Lambda y$$

for all x and $y \in V$ and for all scalars α and β. In the special case in which V_1 is the field of scalars (this is the simplest example of a vector space, except for the trivial one consisting of 0 alone), Λ is called a *linear functional*. A linear functional is thus a complex function on V which satisfies (2).

Note that one often writes Λx, rather than $\Lambda(x)$, if Λ is linear.