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# INTEGRATION ON PRODUCT SPACES

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This chapter is devoted to the proof and discussion of the theorem of Fubini concerning integration of functions of two variables. We first present the theorem in its abstract form.

## Measurability on Cartesian Products

**8.1 Definitions** If  $X$  and  $Y$  are two sets, their *cartesian product*  $X \times Y$  is the set of all ordered pairs  $(x, y)$ , with  $x \in X$  and  $y \in Y$ . If  $A \subset X$  and  $B \subset Y$ , it follows that  $A \times B \subset X \times Y$ . We call any set of the form  $A \times B$  a *rectangle* in  $X \times Y$ .

Suppose now that  $(X, \mathcal{S})$  and  $(Y, \mathcal{T})$  are measurable spaces. Recall that this simply means that  $\mathcal{S}$  is a  $\sigma$ -algebra in  $X$  and  $\mathcal{T}$  is a  $\sigma$ -algebra in  $Y$ .

A *measurable rectangle* is any set of the form  $A \times B$ , where  $A \in \mathcal{S}$  and  $B \in \mathcal{T}$ .

If  $Q = R_1 \cup \cdots \cup R_n$ , where each  $R_i$  is a measurable rectangle and  $R_i \cap R_j = \emptyset$  for  $i \neq j$ , we say that  $Q \in \mathcal{E}$ , the class of all *elementary sets*.

$\mathcal{S} \times \mathcal{T}$  is defined to be the smallest  $\sigma$ -algebra in  $X \times Y$  which contains every measurable rectangle.

A *monotone class*  $\mathfrak{M}$  is a collection of sets with the following properties: If  $A_i \in \mathfrak{M}$ ,  $B_i \in \mathfrak{M}$ ,  $A_i \subset A_{i+1}$ ,  $B_i \supset B_{i+1}$ , for  $i = 1, 2, 3, \dots$ , and if

$$A = \bigcup_{i=1}^{\infty} A_i, \quad B = \bigcap_{i=1}^{\infty} B_i, \quad (1)$$

then  $A \in \mathfrak{M}$  and  $B \in \mathfrak{M}$ .

If  $E \subset X \times Y$ ,  $x \in X$ ,  $y \in Y$ , we define

$$E_x = \{y: (x, y) \in E\}, \quad E^y = \{x: (x, y) \in E\}. \quad (2)$$

We call  $E_x$  and  $E^y$  the  $x$ -section and  $y$ -section, respectively, of  $E$ . Note that  $E_x \subset Y$ ,  $E^y \subset X$ .

**8.2 Theorem** *If  $E \in \mathcal{S} \times \mathcal{T}$ , then  $E_x \in \mathcal{T}$  and  $E^y \in \mathcal{S}$ , for every  $x \in X$  and  $y \in Y$ .*

**PROOF** Let  $\Omega$  be the class of all  $E \in \mathcal{S} \times \mathcal{T}$  such that  $E_x \in \mathcal{T}$  for every  $x \in X$ . If  $E = A \times B$ , then  $E_x = B$  if  $x \in A$ ,  $E_x = \emptyset$  if  $x \notin A$ . Therefore every measurable rectangle belongs to  $\Omega$ . Since  $\mathcal{T}$  is a  $\sigma$ -algebra, the following three statements are true. They prove that  $\Omega$  is a  $\sigma$ -algebra and hence that  $\Omega = \mathcal{S} \times \mathcal{T}$ :

- (a)  $X \times Y \in \Omega$ .
- (b) If  $E \in \Omega$ , then  $(E^c)_x = (E_x)^c$ , hence  $E^c \in \Omega$ .
- (c) If  $E_i \in \Omega$  ( $i = 1, 2, 3, \dots$ ) and  $E = \bigcup E_i$ , then  $E_x = \bigcup (E_i)_x$ , hence  $E \in \Omega$ .

The proof is the same for  $E^y$ .

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**8.3 Theorem**  *$\mathcal{S} \times \mathcal{T}$  is the smallest monotone class which contains all elementary sets.*

**PROOF** Let  $\mathfrak{M}$  be the smallest monotone class which contains  $\mathcal{E}$ ; the proof that this class exists is exactly like that of Theorem 1.10. Since  $\mathcal{S} \times \mathcal{T}$  is a monotone class, we have  $\mathfrak{M} \subset \mathcal{S} \times \mathcal{T}$ .

The identities

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

$$(A_1 \times B_1) - (A_2 \times B_2) = [(A_1 - A_2) \times B_1] \cup [(A_1 \cap A_2) \times (B_1 - B_2)]$$

show that the intersection of two measurable rectangles is a measurable rectangle and that their difference is the union of two disjoint measurable rectangles, hence is an elementary set. If  $P \in \mathcal{E}$  and  $Q \in \mathcal{E}$ , it follows easily that  $P \cap Q \in \mathcal{E}$  and  $P - Q \in \mathcal{E}$ . Since

$$P \cup Q = (P - Q) \cup Q$$

and  $(P - Q) \cap Q = \emptyset$ , we also have  $P \cup Q \in \mathcal{E}$ .

For any set  $P \subset X \times Y$ , define  $\Omega(P)$  to be the class of all  $Q \subset X \times Y$  such that  $P - Q \in \mathfrak{M}$ ,  $Q - P \in \mathfrak{M}$ , and  $P \cup Q \in \mathfrak{M}$ . The following properties are obvious:

- (a)  $Q \in \Omega(P)$  if and only if  $P \in \Omega(Q)$ .
- (b) Since  $\mathfrak{M}$  is a monotone class, so is each  $\Omega(P)$ .

Fix  $P \in \mathcal{E}$ . Our preceding remarks about  $\mathcal{E}$  show that  $Q \in \Omega(P)$  for all  $Q \in \mathcal{E}$ , hence  $\mathcal{E} \subset \Omega(P)$ , and now (b) implies that  $\mathfrak{M} \subset \Omega(P)$ .

Next, fix  $Q \in \mathfrak{M}$ . We just saw that  $Q \in \Omega(P)$  if  $P \in \mathcal{E}$ . By (a),  $P \in \Omega(Q)$ , hence  $\mathcal{E} \subset \Omega(Q)$ , and if we refer to (b) once more we obtain  $\mathfrak{M} \subset \Omega(Q)$ .

Summing up: If  $P$  and  $Q \in \mathfrak{M}$ , then  $P - Q \in \mathfrak{M}$  and  $P \cup Q \in \mathfrak{M}$ .

It now follows that  $\mathfrak{M}$  is a  $\sigma$ -algebra in  $X \times Y$ :

- (i)  $X \times Y \in \mathcal{E}$ . Hence  $X \times Y \in \mathfrak{M}$ .
- (ii) If  $Q \in \mathfrak{M}$ , then  $Q^c \in \mathfrak{M}$ , since the difference of any two members of  $\mathfrak{M}$  is in  $\mathfrak{M}$ .
- (iii) If  $P_i \in \mathfrak{M}$  for  $i = 1, 2, 3, \dots$ , and  $P = \bigcup P_i$ , put

$$Q_n = P_1 \cup \dots \cup P_n.$$

Since  $\mathfrak{M}$  is closed under the formation of finite unions,  $Q_n \in \mathfrak{M}$ .

Since  $Q_n \subset Q_{n+1}$  and  $P = \bigcup Q_n$ , the monotonicity of  $\mathfrak{M}$  shows that  $P \in \mathfrak{M}$ .

Thus  $\mathfrak{M}$  is a  $\sigma$ -algebra,  $\mathcal{E} \subset \mathfrak{M} \subset \mathcal{S} \times \mathcal{T}$ , and (by definition)  $\mathcal{S} \times \mathcal{T}$  is the smallest  $\sigma$ -algebra which contains  $\mathcal{E}$ . Hence  $\mathfrak{M} = \mathcal{S} \times \mathcal{T}$ . ////

**8.4 Definition** With each function  $f$  on  $X \times Y$  and with each  $x \in X$  we associate a function  $f_x$  defined on  $Y$  by  $f_x(y) = f(x, y)$ .

Similarly, if  $y \in Y$ ,  $f^y$  is the function defined on  $X$  by  $f^y(x) = f(x, y)$ .

Since we are now dealing with three  $\sigma$ -algebras,  $\mathcal{S}$ ,  $\mathcal{T}$ , and  $\mathcal{S} \times \mathcal{T}$ , we shall, for the sake of clarity, indicate in the sequel to which of these three  $\sigma$ -algebras the word “measurable” refers.

**8.5 Theorem** Let  $f$  be an  $(\mathcal{S} \times \mathcal{T})$ -measurable function on  $X \times Y$ . Then

- (a) For each  $x \in X$ ,  $f_x$  is a  $\mathcal{T}$ -measurable function.
- (b) For each  $y \in Y$ ,  $f^y$  is an  $\mathcal{S}$ -measurable function.

**PROOF** For any open set  $V$ , put

$$Q = \{(x, y): f(x, y) \in V\}.$$

Then  $Q \in \mathcal{S} \times \mathcal{T}$ , and

$$Q_x = \{y: f_x(y) \in V\}.$$

Theorem 8.2 shows that  $Q_x \in \mathcal{T}$ . This proves (a); the proof of (b) is similar. ////

## Product Measures

**8.6 Theorem** Let  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \lambda)$  be  $\sigma$ -finite measure spaces. Suppose  $Q \in \mathcal{S} \times \mathcal{T}$ . If

$$\varphi(x) = \lambda(Q_x), \quad \psi(y) = \mu(Q^y) \quad (1)$$

for every  $x \in X$  and  $y \in Y$ , then  $\varphi$  is  $\mathcal{S}$ -measurable,  $\psi$  is  $\mathcal{T}$ -measurable, and

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda, \quad (2)$$

*Notes:* The assumptions on the measure spaces are, more explicitly, that  $\mu$  and  $\lambda$  are positive measures on  $\mathcal{S}$  and  $\mathcal{T}$ , respectively, that  $X$  is the union of countably many disjoint sets  $X_n$  with  $\mu(X_n) < \infty$ , and that  $Y$  is the union of countably many disjoint sets  $Y_m$  with  $\lambda(Y_m) < \infty$ .

Theorem 8.2 shows that the definitions (1) make sense. Since

$$\lambda(Q_x) = \int_Y \chi_Q(x, y) \, d\lambda(y) \quad (x \in X), \quad (3)$$

with a similar statement for  $\mu(Q^y)$ , the conclusion (2) can be written in the form

$$\int_X d\mu(x) \int_Y \chi_Q(x, y) \, d\lambda(y) = \int_Y d\lambda(y) \int_X \chi_Q(x, y) \, d\mu(x). \quad (4)$$

**PROOF** Let  $\Omega$  be the class of all  $Q \in \mathcal{S} \times \mathcal{T}$  for which the conclusion of the theorem holds. We claim that  $\Omega$  has the following four properties:

- (a) Every measurable rectangle belongs to  $\Omega$ .
- (b) If  $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$ , if each  $Q_i \in \Omega$ , and if  $Q = \bigcup Q_i$ , then  $Q \in \Omega$ .
- (c) If  $\{Q_i\}$  is a disjoint countable collection of members of  $\Omega$ , and if  $Q = \bigcup Q_i$ , then  $Q \in \Omega$ .
- (d) If  $\mu(A) < \infty$  and  $\lambda(B) < \infty$ , if  $A \times B \supset Q_1 \supset Q_2 \supset Q_3 \supset \cdots$ , if  $Q = \bigcap Q_i$  and  $Q_i \in \Omega$  for  $i = 1, 2, 3, \dots$ , then  $Q \in \Omega$ .

If  $Q = A \times B$ , where  $A \in \mathcal{S}$ ,  $B \in \mathcal{T}$ , then

$$\lambda(Q_x) = \lambda(B)\chi_A(x) \quad \text{and} \quad \mu(Q^y) = \mu(A)\chi_B(y), \quad (5)$$

and therefore each of the integrals in (2) is equal to  $\mu(A)\lambda(B)$ . This gives (a).

To prove (b), let  $\varphi_i$  and  $\psi_i$  be associated with  $Q_i$  in the way in which (1) associates  $\varphi$  and  $\psi$  with  $Q$ . The countable additivity of  $\mu$  and  $\lambda$  shows that

$$\varphi_i(x) \rightarrow \varphi(x), \quad \psi_i(y) \rightarrow \psi(y) \quad (i \rightarrow \infty), \quad (6)$$

the convergence being monotone increasing at every point. Since  $\varphi_i$  and  $\psi_i$  are assumed to satisfy the conclusion of the theorem, (b) follows from the monotone convergence theorem.

For finite unions of disjoint sets, (c) is clear, because the characteristic function of a union of *disjoint* sets is the sum of their characteristic functions. The general case of (c) now follows from (b).

The proof of (d) is like that of (b), except that we use the dominated convergence theorem in place of the monotone convergence theorem. This is legitimate, since  $\mu(A) < \infty$  and  $\lambda(B) < \infty$ .

Now define

$$Q_{mn} = Q \cap (X_n \times Y_m) \quad (m, n = 1, 2, 3, \dots) \quad (7)$$

and let  $\mathfrak{M}$  be the class of all  $Q \in \mathcal{S} \times \mathcal{T}$  such that  $Q_{mn} \in \Omega$  for all choices of  $m$  and  $n$ . Then (b) and (d) show that  $\mathfrak{M}$  is a monotone class; (a) and (c) show that  $\mathcal{E} \subset \mathfrak{M}$ ; and since  $\mathfrak{M} \subset \mathcal{S} \times \mathcal{T}$ , Theorem 8.3 implies that  $\mathfrak{M} = \mathcal{S} \times \mathcal{T}$ .

Thus  $Q_{mn} \in \Omega$  for every  $Q \in \mathcal{S} \times \mathcal{T}$  and for all choices of  $m$  and  $n$ . Since  $Q$  is the union of the sets  $Q_{mn}$  and since these sets are disjoint, we conclude from (c) that  $Q \in \Omega$ . This completes the proof. ////

**8.7 Definition** If  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \lambda)$  are as in Theorem 8.6, and if  $Q \in \mathcal{S} \times \mathcal{T}$ , we define

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x) d\mu(x) = \int_Y \mu(Q^y) d\lambda(y). \quad (1)$$

The equality of the integrals in (1) is the content of Theorem 8.6. We call  $\mu \times \lambda$  the *product* of the measures  $\mu$  and  $\lambda$ . That  $\mu \times \lambda$  is really a measure (i.e., that  $\mu \times \lambda$  is countably additive on  $\mathcal{S} \times \mathcal{T}$ ) follows immediately from Theorem 1.27.

Observe also that  $\mu \times \lambda$  is  $\sigma$ -finite.

## The Fubini Theorem

**8.8 Theorem** Let  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \lambda)$  be  $\sigma$ -finite measure spaces, and let  $f$  be an  $(\mathcal{S} \times \mathcal{T})$ -measurable function on  $X \times Y$ .

(a) If  $0 \leq f \leq \infty$ , and if

$$\varphi(x) = \int_Y f_x d\lambda, \quad \psi(y) = \int_X f^y d\mu \quad (x \in X, y \in Y), \quad (1)$$

then  $\varphi$  is  $\mathcal{S}$ -measurable,  $\psi$  is  $\mathcal{T}$ -measurable, and

$$\int_X \varphi d\mu = \int_{X \times Y} f d(\mu \times \lambda) = \int_Y \psi d\lambda. \quad (2)$$

(b) If  $f$  is complex and if

$$\varphi^*(x) = \int_Y |f_x| d\lambda \quad \text{and} \quad \int_X \varphi^* d\mu < \infty, \quad (3)$$

then  $f \in L^1(\mu \times \lambda)$ .

- (c) If  $f \in L^1(\mu \times \lambda)$ , then  $f_x \in L^1(\lambda)$  for almost all  $x \in X$ ,  $f_y \in L^1(\mu)$  for almost all  $y \in Y$ ; the functions  $\varphi$  and  $\psi$ , defined by (1) a.e., are in  $L^1(\mu)$  and  $L^1(\lambda)$ , respectively, and (2) holds.

Notes: The first and last integrals in (2) can also be written in the more usual form

$$\int_X d\mu(x) \int_Y f(x, y) d\lambda(y) = \int_Y d\lambda(y) \int_X f(x, y) d\mu(x). \quad (4)$$

These are the so-called "iterated integrals" of  $f$ . The middle integral in (2) is often referred to as a *double integral*.

The combination of (b) and (c) gives the following useful result: If  $f$  is  $(\mathcal{S} \times \mathcal{T})$ -measurable and if

$$\int_X d\mu(x) \int_Y |f(x, y)| d\lambda(y) < \infty, \quad (5)$$

then the two iterated integrals (4) are finite and equal.

In other words, "the order of integration may be reversed" for  $(\mathcal{S} \times \mathcal{T})$ -measurable functions  $f$  whenever  $f \geq 0$  and also whenever one of the iterated integrals of  $|f|$  is finite.

PROOF We first consider (a). By Theorem 8.5, the definitions of  $\varphi$  and  $\psi$  make sense. Suppose  $Q \in \mathcal{S} \times \mathcal{T}$  and  $f = \chi_Q$ . By Definition 8.7, (2) is then exactly the conclusion of Theorem 8.6. Hence (a) holds for all nonnegative simple  $(\mathcal{S} \times \mathcal{T})$ -measurable functions  $s$ . In the general case, there is a sequence of such functions  $s_n$ , such that  $0 \leq s_1 \leq s_2 \leq \dots$  and  $s_n(x, y) \rightarrow f(x, y)$  at every point of  $X \times Y$ . If  $\varphi_n$  is associated with  $s_n$  in the same way in which  $\varphi$  was associated to  $f$ , we have

$$\int_X \varphi_n d\mu = \int_{X \times Y} s_n d(\mu \times \lambda) \quad (n = 1, 2, 3, \dots). \quad (6)$$

The monotone convergence theorem, applied on  $(Y, \mathcal{T}, \lambda)$ , shows that  $\varphi_n(x)$  increases to  $\varphi(x)$ , for every  $x \in X$ , as  $n \rightarrow \infty$ . Hence the monotone convergence theorem applies again, to the two integrals in (6), and the first equality (2) is obtained. The second half of (2) follows by interchanging the roles of  $x$  and  $y$ . This completes (a).

If we apply (a) to  $|f|$ , we see that (b) is true.

Obviously, it is enough to prove (c) for real  $L^1(\mu \times \lambda)$ ; the complex case then follows. If  $f$  is real, (a) applies to  $f^+$  and to  $f^-$ . Let  $\varphi_1$  and  $\varphi_2$  correspond to  $f^+$  and  $f^-$  as  $\varphi$  corresponds to  $f$  in (1). Since  $f \in L^1(\mu \times \lambda)$  and

$f^+ \leq |f|$ , and since (a) holds for  $f^+$ , we see that  $\varphi_1 \in L^1(\mu)$ . Similarly,  $\varphi_2 \in L^1(\mu)$ . Since

$$f_x = (f^+)_x - (f^-)_x \quad (7)$$

we have  $f_x \in L^1(\lambda)$  for every  $x$  for which  $\varphi_1(x) < \infty$  and  $\varphi_2(x) < \infty$ ; since  $\varphi_1$  and  $\varphi_2$  are in  $L^1(\mu)$ , this happens for almost all  $x$ ; and at any such  $x$ , we have  $\varphi(x) = \varphi_1(x) - \varphi_2(x)$ . Hence  $\varphi \in L^1(\mu)$ . Now (2) holds with  $\varphi_1$  and  $f^+$  and with  $\varphi_2$  and  $f^-$ , in place of  $\varphi$  and  $f$ ; if we subtract the resulting equations, we obtain one half of (c). The other half is proved in the same manner, with  $f^y$  and  $\psi$  in place of  $f_x$  and  $\varphi$ . ////

**8.9 Counterexamples** The following three examples will show that the various hypotheses in Theorems 8.6 and 8.8 cannot be dispensed with.

- (a) Let  $X = Y = [0, 1]$ ,  $\mu = \lambda =$  Lebesgue measure on  $[0, 1]$ . Choose  $\{\delta_n\}$  so that  $0 = \delta_1 < \delta_2 < \delta_3 < \dots$ ,  $\delta_n \rightarrow 1$ , and let  $g_n$  be a real continuous function with support in  $(\delta_n, \delta_{n+1})$ , such that  $\int_0^1 g_n(t) dt = 1$ , for  $n = 1, 2, 3, \dots$ . Define

$$f(x, y) = \sum_{n=1}^{\infty} [g_n(x) - g_{n+1}(x)]g_n(y).$$

Note that at each point  $(x, y)$  at most one term in this sum is different from 0. Thus no convergence problem arises in the definition of  $f$ . An easy computation shows that

$$\int_0^1 dx \int_0^1 f(x, y) dy = 1 \neq 0 = \int_0^1 dy \int_0^1 f(x, y) dx,$$

so that the conclusion of the Fubini theorem fails, although both iterated integrals exist. Note that  $f$  is continuous in this example, except at the point  $(1, 1)$ , but that

$$\int_0^1 dx \int_0^1 |f(x, y)| dy = \infty.$$

- (b) Let  $X = Y = [0, 1]$ ,  $\mu =$  Lebesgue measure on  $[0, 1]$ ,  $\lambda =$  counting measure on  $Y$ , and put  $f(x, y) = 1$  if  $x = y$ ,  $f(x, y) = 0$  if  $x \neq y$ . Then

$$\int_X f(x, y) d\mu(x) = 0, \quad \int_Y f(x, y) d\lambda(y) = 1$$

for all  $x$  and  $y$  in  $[0, 1]$ , so that

$$\int_Y d\lambda(y) \int_X f(x, y) d\mu(x) = 0 \neq 1 = \int_X d\mu(x) \int_Y f(x, y) d\lambda(y).$$

This time the failure is due to the fact that  $\lambda$  is not  $\sigma$ -finite.

Observe that our function  $f$  is  $(\mathcal{S} \times \mathcal{T})$ -measurable, if  $\mathcal{S}$  is the class of all Lebesgue measurable sets in  $[0, 1]$  and  $\mathcal{T}$  consists of all subsets of  $[0, 1]$ .

To see this, note that  $f = \chi_D$ , where  $D$  is the diagonal of the unit square. Given  $n$ , put

$$I_j = \left[ \frac{j-1}{n}, \frac{j}{n} \right]$$

and put

$$Q_n = (I_1 \times I_1) \cup (I_2 \times I_2) \cup \cdots \cup (I_n \times I_n).$$

Then  $Q_n$  is a finite union of measurable rectangles, and  $D = \bigcap Q_n$ .

- (c) In examples (a) and (b), the failure of the Fubini theorem was due to the fact that either the function or the space was “too big.” We now turn to the role played by the requirement that  $f$  be measurable with respect to the  $\sigma$ -algebra  $\mathcal{S} \times \mathcal{T}$ .

To pose the question more precisely, suppose  $\mu(X) = \lambda(Y) = 1$ ,  $0 \leq f \leq 1$  (so that “bigness” is certainly avoided); assume  $f_x$  is  $\mathcal{T}$ -measurable and  $f^y$  is  $\mathcal{S}$ -measurable, for all  $x$  and  $y$ ; and assume  $\varphi$  is  $\mathcal{S}$ -measurable and  $\psi$  is  $\mathcal{T}$ -measurable, where  $\varphi$  and  $\psi$  are defined as in 8.8(1). Then  $0 \leq \varphi \leq 1$ ,  $0 \leq \psi \leq 1$ , and both iterated integrals are finite. (Note that no reference to product measures is needed to *define* iterated integrals.) Does it follow that the two iterated integrals of  $f$  are equal?

The (perhaps surprising) answer is no.

In the following example (due to Sierpinski), we take

$$(X, \mathcal{S}, \mu) = (Y, \mathcal{T}, \lambda) = [0, 1]$$

with Lebesgue measure. The construction depends on the continuum hypothesis. It is a consequence of this hypothesis that there is a one-to-one mapping  $j$  of the unit interval  $[0, 1]$  onto a well-ordered set  $W$  such that  $j(x)$  has at most countably many predecessors in  $W$ , for each  $x \in [0, 1]$ . Taking this for granted, let  $Q$  be the set of all  $(x, y)$  in the unit square such that  $j(x)$  precedes  $j(y)$  in  $W$ . For each  $x \in [0, 1]$ ,  $Q_x$  contains all but countably many points of  $[0, 1]$ ; for each  $y \in [0, 1]$ ,  $Q^y$  contains at most countably many points of  $[0, 1]$ . If  $f = \chi_Q$ , it follows that  $f_x$  and  $f^y$  are Borel measurable and that

$$\varphi(x) = \int_0^1 f(x, y) dy = 1, \quad \psi(y) = \int_0^1 f(x, y) dx = 0$$

for all  $x$  and  $y$ . Hence

$$\int_0^1 dx \int_0^1 f(x, y) dy = 1 \neq 0 = \int_0^1 dy \int_0^1 f(x, y) dx.$$

## Completion of Product Measures

**8.10** If  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \lambda)$  are complete measure spaces, it need not be true that  $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times \lambda)$  is complete. There is nothing pathological about



this phenomenon: Suppose that there exists an  $A \in \mathcal{S}$ ,  $A \neq \emptyset$ , with  $\mu(A) = 0$ ; and suppose that there exists a  $B \subset Y$  so that  $B \notin \mathcal{T}$ . Then  $A \times B \subset A \times Y$ ,  $(\mu \times \lambda)(A \times Y) = 0$ , but  $A \times B \notin \mathcal{S} \times \mathcal{T}$ . (The last assertion follows from Theorem 8.2.)

For instance, if  $\mu = \lambda = m_1$  (Lebesgue measure on  $R^1$ ), let  $A$  consist of any one point, and let  $B$  be any nonmeasurable set in  $R^1$ . Thus  $m_1 \times m_1$  is not a complete measure; in particular,  $m_1 \times m_1$  is not  $m_2$ , since the latter is complete, by its construction. However,  $m_2$  is the completion of  $m_1 \times m_1$ . This result generalizes to arbitrary dimensions:

**8.11 Theorem** *Let  $m_k$  denote Lebesgue measure on  $R^k$ . If  $k = r + s$ ,  $r \geq 1$ ,  $s \geq 1$ , then  $m_k$  is the completion of the product measure  $m_r \times m_s$ .*

PROOF Let  $\mathcal{B}_k$  and  $\mathcal{M}_k$  be the  $\sigma$ -algebras of all Borel sets and of all Lebesgue measurable sets in  $R^k$ , respectively. We shall first show that

$$\mathcal{B}_k \subset \mathcal{M}_r \times \mathcal{M}_s \subset \mathcal{M}_k. \quad (1)$$

Every  $k$ -cell belongs to  $\mathcal{M}_r \times \mathcal{M}_s$ . The  $\sigma$ -algebra generated by the  $k$ -cells is  $\mathcal{B}_k$ . Hence  $\mathcal{B}_k \subset \mathcal{M}_r \times \mathcal{M}_s$ . Next, suppose  $E \in \mathcal{M}_r$  and  $F \in \mathcal{M}_s$ . It is easy to see, by Theorem 2.20(b), that both  $E \times R^s$  and  $R^r \times F$  belong to  $\mathcal{M}_k$ . The same is true of their intersection  $E \times F$ . It follows that  $\mathcal{M}_r \times \mathcal{M}_s \subset \mathcal{M}_k$ .

Choose  $Q \in \mathcal{M}_r \times \mathcal{M}_s$ . Then  $Q \in \mathcal{M}_k$ , so there are sets  $P_1$  and  $P_2 \in \mathcal{B}_k$  such that  $P_1 \subset Q \subset P_2$  and  $m_k(P_2 - P_1) = 0$ . Both  $m_k$  and  $m_r \times m_s$  are translation invariant Borel measures on  $R^k$ . They assign the same value to each  $k$ -cell. Hence they agree on  $\mathcal{B}_k$ , by Theorem 2.20(d). In particular,

$$(m_r \times m_s)(Q - P_1) \leq (m_r \times m_s)(P_2 - P_1) = m_k(P_2 - P_1) = 0$$

and therefore

$$(m_r \times m_s)(Q) = (m_r \times m_s)(P_1) = m_k(P_1) = m_k(Q).$$

So  $m_r \times m_s$  agrees with  $m_k$  on  $\mathcal{M}_r \times \mathcal{M}_s$ .

It now follows that  $\mathcal{M}_k$  is the  $(m_r \times m_s)$ -completion of  $\mathcal{M}_r \times \mathcal{M}_s$ , and this is what the theorem asserts. ////

We conclude this section with an alternative statement of Fubini's theorem which is of special interest in view of Theorem 8.11.

**8.12 Theorem** *Let  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \lambda)$  be complete  $\sigma$ -finite measure spaces. Let  $(\mathcal{S} \times \mathcal{T})^*$  be the completion of  $\mathcal{S} \times \mathcal{T}$ , relative to the measure  $\mu \times \lambda$ . Let  $f$  be an  $(\mathcal{S} \times \mathcal{T})^*$ -measurable function on  $X \times Y$ . Then all conclusions of Theorem 8.8 hold, the only difference being as follows:*

*The  $\mathcal{T}$ -measurability of  $f_x$  can be asserted only for almost all  $x \in X$ , so that  $\varphi(x)$  is only defined a.e.  $[\mu]$  by 8.8(1); a similar statement holds for  $f^y$  and  $\psi$ .*

The proof depends on the following two lemmas:

**Lemma 1** Suppose  $v$  is a positive measure on a  $\sigma$ -algebra  $\mathfrak{M}$ ,  $\mathfrak{M}^*$  is the completion of  $\mathfrak{M}$  relative to  $v$ , and  $f$  is an  $\mathfrak{M}^*$ -measurable function. Then there exists an  $\mathfrak{M}$ -measurable function  $g$  such that  $f = g$  a.e.  $[v]$ .

(An interesting special case of this arises when  $v$  is Lebesgue measure on  $R^k$  and  $\mathfrak{M}$  is the class of all Borel sets in  $R^k$ .)

**Lemma 2** Let  $h$  be an  $(\mathcal{S} \times \mathcal{T})^*$ -measurable function on  $X \times Y$  such that  $h = 0$  a.e.  $[\mu \times \lambda]$ . Then for almost all  $x \in X$  it is true that  $h(x, y) = 0$  for almost all  $y \in Y$ ; in particular,  $h_x$  is  $\mathcal{T}$ -measurable for almost all  $x \in X$ . A similar statement holds for  $h^y$ .

If we assume the lemmas, the proof of the theorem is immediate: If  $f$  is as in the theorem, Lemma 1 (with  $v = \mu \times \lambda$ ) shows that  $f = g + h$ , where  $h = 0$  a.e.  $[\mu \times \lambda]$  and  $g$  is  $(\mathcal{S} \times \mathcal{T})$ -measurable. Theorem 8.8 applies to  $g$ . Lemma 2 shows that  $f_x = g_x$  a.e.  $[\lambda]$  for almost all  $x$  and that  $f^y = g^y$  a.e.  $[\mu]$  for almost all  $y$ . Hence the two iterated integrals of  $f$ , as well as the double integral, are the same as those of  $g$ , and the theorem follows.

**PROOF OF LEMMA 1** Suppose  $f$  is  $\mathfrak{M}^*$ -measurable and  $f \geq 0$ . There exist  $\mathfrak{M}^*$ -measurable simple functions  $0 = s_0 \leq s_1 \leq s_2 \leq \dots$  such that  $s_n(x) \rightarrow f(x)$  for each  $x \in X$ , as  $n \rightarrow \infty$ . Hence  $f = \sum (s_{n+1} - s_n)$ . Since  $s_{n+1} - s_n$  is a finite linear combination of characteristic functions, it follows that there are constants  $c_i > 0$  and sets  $E_i \in \mathfrak{M}^*$  such that

$$f(x) = \sum_{i=1}^{\infty} c_i \chi_{E_i}(x) \quad (x \in X).$$

The definition of  $\mathfrak{M}^*$  (see Theorem 1.36) shows that there are sets  $A_i \in \mathfrak{M}$ ,  $B_i \in \mathfrak{M}$ , such that  $A_i \subset E_i \subset B_i$  and  $v(B_i - A_i) = 0$ . Define

$$g(x) = \sum_{i=1}^{\infty} c_i \chi_{A_i}(x) \quad (x \in X).$$

Then the function  $g$  is  $\mathfrak{M}$ -measurable, and  $g(x) = f(x)$ , except possibly when  $x \in \bigcup (E_i - A_i) \subset \bigcup (B_i - A_i)$ . Since  $v(B_i - A_i) = 0$  for each  $i$ , we conclude that  $g = f$  a.e.  $[v]$ . The general case ( $f$  real or complex) follows from this. ///

**PROOF OF LEMMA 2** Let  $P$  be the set of all points in  $X \times Y$  at which  $h(x, y) \neq 0$ . Then  $P \in (\mathcal{S} \times \mathcal{T})^*$  and  $(\mu \times \lambda)(P) = 0$ . Hence there exists a  $Q \in \mathcal{S} \times \mathcal{T}$  such that  $P \subset Q$  and  $(\mu \times \lambda)(Q) = 0$ . By Theorem 8.6,

$$\int_X \lambda(Q_x) d\mu(x) = 0. \quad (1)$$

Let  $N$  be the set of all  $x \in X$  at which  $\lambda(Q_x) > 0$ . It follows from (1) that  $\mu(N) = 0$ . For every  $x \notin N$ ,  $\lambda(Q_x) = 0$ . Since  $P_x \subset Q_x$  and  $(Y, \mathcal{T}, \lambda)$  is a complete measure space, every subset of  $P_x$  belongs to  $\mathcal{T}$  if  $x \notin N$ . If  $y \notin P_x$ , then  $h_x(y) = 0$ . Thus we see, for every  $x \notin N$ , that  $h_x$  is  $\mathcal{T}$ -measurable and that  $h_x(y) = 0$  a.e.  $[\lambda]$ . ////

## Convolutions

**8.13** It happens occasionally that one can prove that a certain set is not empty by proving that it is actually large. The word “large” may of course refer to various properties. One of these (a rather crude one) is cardinality. An example is furnished by the familiar proof that there exist transcendental numbers: There are only countably many algebraic numbers but uncountably many real numbers, hence the set of transcendental real numbers is not empty. Applications of Baire’s theorem are based on a topological notion of largeness: The dense  $G_\delta$ ’s are “large” subsets of a complete metric space. A third type of largeness is measure-theoretic: One can try to show that a certain set in a measure space is not empty by showing that it has positive measure or, better still, by showing that its complement has measure zero. Fubini’s theorem often occurs in this type of argument.

For example, let  $f$  and  $g \in L^1(\mathbb{R}^1)$ , assume  $f \geq 0$  and  $g \geq 0$  for the moment, and consider the integral

$$h(x) = \int_{-\infty}^{\infty} f(x-t)g(t) dt \quad (-\infty < x < \infty). \quad (1)$$

For any fixed  $x$ , the integrand in (1) is a measurable function with range in  $[0, \infty]$ , so that  $h(x)$  is certainly well defined by (1), and  $0 \leq h(x) \leq \infty$ .

But is there any  $x$  for which  $h(x) < \infty$ ? Note that the integrand in (1) is, for each fixed  $x$ , the product of two members of  $L^1$ , and such a product is not always in  $L^1$ . [Example:  $f(x) = g(x) = 1/\sqrt{x}$  if  $0 < x < 1$ , 0 otherwise.] The Fubini theorem will give an affirmative answer. In fact, it will show that  $h \in L^1(\mathbb{R}^1)$ , hence that  $h(x) < \infty$  a.e.

**8.14 Theorem** Suppose  $f \in L^1(\mathbb{R}^1)$ ,  $g \in L^1(\mathbb{R}^1)$ . Then

$$\int_{-\infty}^{\infty} |f(x-y)g(y)| dy < \infty \quad (1)$$

for almost all  $x$ . For these  $x$ , define

$$h(x) = \int_{-\infty}^{\infty} f(x-y)g(y) dy. \quad (2)$$

Then  $h \in L^1(\mathbb{R}^1)$ , and

$$\|h\|_1 \leq \|f\|_1 \|g\|_1, \quad (3)$$

where

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)| dx. \quad (4)$$

We call  $h$  the *convolution* of  $f$  and  $g$ , and write  $h = f * g$ .

**PROOF** There exist Borel functions  $f_0$  and  $g_0$  such that  $f_0 = f$  a.e. and  $g_0 = g$  a.e. The integrals (1) and (2) are unchanged, for every  $x$ , if we replace  $f$  by  $f_0$  and  $g$  by  $g_0$ . Hence we may assume, to begin with, that  $f$  and  $g$  are Borel functions.

To apply Fubini's theorem, we shall first prove that the function  $F$  defined by

$$F(x, y) = f(x - y)g(y) \quad (5)$$

is a Borel function on  $R^2$ .

Define  $\varphi: R^2 \rightarrow R^1$  and  $\psi: R^2 \rightarrow R^1$  by

$$\varphi(x, y) = x - y, \quad \psi(x, y) = y. \quad (6)$$

Then  $f(x - y) = (f \circ \varphi)(x, y)$  and  $g(y) = (g \circ \psi)(x, y)$ . Since  $\varphi$  and  $\psi$  are Borel functions, Theorem 1.12(d) shows that  $f \circ \varphi$  and  $g \circ \psi$  are Borel functions on  $R^2$ . Hence so is their product.

Next we observe that

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |F(x, y)| dx = \int_{-\infty}^{\infty} |g(y)| dy \int_{-\infty}^{\infty} |f(x - y)| dx = \|f\|_1 \|g\|_1, \quad (7)$$

$$\text{since} \quad \int_{-\infty}^{\infty} |f(x - y)| dx = \|f\|_1 \quad (8)$$

for every  $y \in R^1$ , by the translation-invariance of Lebesgue measure.

Thus  $F \in L^1(R^2)$ , and Fubini's theorem implies that the integral (2) exists for almost all  $x \in R^1$  and that  $h \in L^1(R^1)$ . Finally,

$$\begin{aligned} \|h\|_1 &= \int_{-\infty}^{\infty} |h(x)| dx \leq \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |F(x, y)| dy \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |F(x, y)| dx = \|f\|_1 \|g\|_1, \end{aligned}$$

by (7). This gives (3), and completes the proof. ////

Convolutions will play an important role in Chap. 9.

## Distribution Functions

**8.15 Definition** Let  $\mu$  be a  $\sigma$ -finite positive measure on some  $\sigma$ -algebra in some set  $X$ . Let  $f: X \rightarrow [0, \infty]$  be measurable. The function that assigns to each  $t \in [0, \infty)$  the number

$$\mu\{f > t\} = \mu(\{x \in X: f(x) > t\}) \quad (1)$$

is called the *distribution function* of  $f$ . It is clearly a monotonic (nonincreasing) function of  $t$  and is therefore Borel measurable.

One reason for introducing distribution functions is that they make it possible to replace integrals over  $X$  by integrals over  $[0, \infty)$ ; the formula

$$\int_X f \, d\mu = \int_0^\infty \mu\{f > t\} \, dt \quad (2)$$

is the special case  $\varphi(t) = t$  of our next theorem. This will then be used to derive an  $L^p$ -property of the maximal functions that were introduced in Chap. 7.

**8.16 Theorem** Suppose that  $f$  and  $\mu$  are as above, that  $\varphi: [0, \infty) \rightarrow [0, \infty]$  is monotonic, absolutely continuous on  $[0, T]$  for every  $T < \infty$ , and that  $\varphi(0) = 0$  and  $\varphi(t) \rightarrow \varphi(\infty)$  as  $t \rightarrow \infty$ . Then

$$\int_X (\varphi \circ f) \, d\mu = \int_0^\infty \mu\{f > t\} \varphi'(t) \, dt. \quad (1)$$

**PROOF** Let  $E$  be the set of all  $(x, t) \in X \times [0, \infty)$  where  $f(x) > t$ . When  $f$  is simple, then  $E$  is a union of finitely many measurable rectangles, and is therefore measurable. In the general case, the measurability of  $E$  follows via the standard approximation of  $f$  by simple functions (Theorem 1.17). As in Sec. 8.1, put

$$E^t = \{x \in X: (x, t) \in E\} \quad (0 \leq t < \infty). \quad (2)$$

The distribution function of  $f$  is then

$$\mu(E^t) = \int_X \chi_E(x, t) \, d\mu(x). \quad (3)$$

The right side of (1) is therefore

$$\int_0^\infty \mu(E^t) \varphi'(t) \, dt = \int_X d\mu(x) \int_0^\infty \chi_E(x, t) \varphi'(t) \, dt, \quad (4)$$

by Fubini's theorem.

For each  $x \in X$ ,  $\chi_E(x, t) = 1$  if  $t < f(x)$  and is 0 if  $t \geq f(x)$ . The inner integral in (4) is therefore

$$\int_0^{f(x)} \varphi'(t) dt = \varphi(f(x)) \quad (5)$$

by Theorem 7.20. Now (1) follows from (4) and (5). ////

**8.17** Recall now that the maximal function  $Mf$  lies in weak  $L^1$  when  $f \in L^1(R^k)$  (Theorem 7.4). We also have the trivial estimate

$$\|Mf\|_\infty \leq \|f\|_\infty \quad (1)$$

valid for all  $f \in L^\infty(R^k)$ . A technique invented by Marcinkiewicz makes it possible to “interpolate” between these two extremes and to prove the following theorem of Hardy and Littlewood (which fails when  $p = 1$ ; see Exercise 22, Chap. 7).

**8.18 Theorem** *If  $1 < p < \infty$  and  $f \in L^p(R^k)$  then  $Mf \in L^p(R^k)$ .*

**PROOF** Since  $Mf = M(|f|)$  we may assume, without loss of generality, that  $f \geq 0$ . Theorem 7.4 shows that there is a constant  $A$ , depending only on the dimension  $k$ , such that

$$m\{Mg > t\} \leq \frac{A}{t} \|g\|_1 \quad (1)$$

for every  $g \in L^1(R^k)$ . Here, and in the rest of this proof,  $m = m_k$ , the Lebesgue measure on  $R^k$ .

Pick a constant  $c$ ,  $0 < c < 1$ , which will be specified later so as to minimize a certain upper bound. For each  $t \in (0, \infty)$ , split  $f$  into a sum

$$f = g_t + h_t \quad (2)$$

where

$$g_t(x) = \begin{cases} f(x) & \text{if } f(x) > ct \\ 0 & \text{if } f(x) \leq ct. \end{cases} \quad (3)$$

Then  $0 \leq h_t(x) \leq ct$  for every  $x \in R^k$ . Hence  $h_t \in L^\infty$ ,  $Mh_t \leq ct$ , and

$$Mf \leq Mg_t + Mh_t \leq Mg_t + ct. \quad (4)$$

If  $(Mf)(x) > t$  for some  $x$ , it follows that

$$(Mg_t)(x) > (1 - c)t. \quad (5)$$

Setting  $E_t = \{f > ct\}$ , (5), (1), and (3) imply that

$$m\{Mf > t\} \leq m\{Mg_t > (1 - c)t\} \leq \frac{A}{(1 - c)t} \|g_t\|_1 = \frac{A}{(1 - c)t} \int_{E_t} f dm.$$

We now use Theorem 8.16, with  $X = R^k$ ,  $\mu = m$ ,  $\varphi(t) = t^p$ , to calculate

$$\begin{aligned} \int_{R^k} (Mf)^p dm &= p \int_0^\infty m\{Mf > t\} t^{p-1} dt \leq \frac{Ap}{1-c} \int_0^\infty t^{p-2} dt \int_{E_t} f dm \\ &= \frac{Ap}{1-c} \int_{R^k} f dm \int_0^{f/c} t^{p-2} dt = \frac{Apc^{1-p}}{(1-c)(p-1)} \int_{R^k} f^p dm. \end{aligned}$$

This proves the theorem. However, to get a good constant, let us choose  $c$  so as to minimize that last expression. This happens when  $c = (p-1)/p = 1/q$ , where  $q$  is the exponent conjugate to  $p$ . For this  $c$ ,

$$c^{1-p} = \left(1 + \frac{1}{p-1}\right)^{p-1} < e,$$

and the preceding computation yields

$$\|Mf\|_p \leq C_p \|f\|_p \quad (6)$$

where  $C_p = (Aepq)^{1/p}$ .

////

Note that  $C_p \rightarrow 1$  as  $p \rightarrow \infty$ , which agrees with formula 8.17(1), and that  $C_p \rightarrow \infty$  as  $p \rightarrow 1$ .

## Exercises

1 Find a monotone class  $\mathfrak{M}$  in  $R^1$  which is not a  $\sigma$ -algebra, even though  $R^1 \in \mathfrak{M}$  and  $R^1 - A \in \mathfrak{M}$  for every  $A \in \mathfrak{M}$ .

2 Suppose  $f$  is a Lebesgue measurable nonnegative real function on  $R^1$  and  $A(f)$  is the *ordinate set* of  $f$ . This is the set of all points  $(x, y) \in R^2$  for which  $0 < y < f(x)$ .

- Is it true that  $A(f)$  is Lebesgue measurable, in the two-dimensional sense?
- If the answer to (a) is affirmative, is the integral of  $f$  over  $R^1$  equal to the measure of  $A(f)$ ?
- Is the graph of  $f$  a measurable subset of  $R^2$ ?
- If the answer to (c) is affirmative, is the measure of the graph equal to zero?

3 Find an example of a positive continuous function  $f$  in the open unit square in  $R^2$ , whose integral (relative to Lebesgue measure) is finite but such that  $\varphi(x)$  (in the notation of Theorem 8.8) is infinite for some  $x \in (0, 1)$ .

4 Suppose  $1 \leq p \leq \infty$ ,  $f \in L^1(R^1)$ , and  $g \in L^p(R^1)$ .

(a) Imitate the proof of Theorem 8.14 to show that the integral defining  $(f * g)(x)$  exists for almost all  $x$ , that  $f * g \in L^p(R^1)$ , and that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

(b) Show that equality can hold in (a) if  $p = 1$  and if  $p = \infty$ , and find the conditions under which this happens.

(c) Assume  $1 < p < \infty$ , and equality holds in (a). Show that then either  $f = 0$  a.e. or  $g = 0$  a.e.

(d) Assume  $1 \leq p \leq \infty$ ,  $\epsilon > 0$ , and show that there exist  $f \in L^1(R^1)$  and  $g \in L^p(R^1)$  such that

$$\|f * g\|_p > (1 - \epsilon) \|f\|_1 \|g\|_p.$$

5 Let  $M$  be the Banach space of all complex Borel measures on  $R^1$ . The norm in  $M$  is  $\|\mu\| = |\mu|(R^1)$ . Associate to each Borel set  $E \subset R^1$  the set

$$E_2 = \{(x, y): x + y \in E\} \subset R^2.$$

If  $\mu$  and  $\lambda \in M$ , define their convolution  $\mu * \lambda$  to be the set function given by

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_2)$$

for every Borel set  $E \subset R^1$ ;  $\mu * \lambda$  is as in Definition 8.7.

(a) Prove that  $\mu * \lambda \in M$  and that  $\|\mu * \lambda\| \leq \|\mu\| \|\lambda\|$ .

(b) Prove that  $\mu * \lambda$  is the unique  $\nu \in M$  such that

$$\int f d\nu = \iint f(x + y) d\mu(x) d\lambda(y)$$

for every  $f \in C_0(R^1)$ . (All integrals extend over  $R^1$ .)

(c) Prove that convolution in  $M$  is commutative, associative, and distributive with respect to addition.

Prove the formula

$$(\mu * \lambda)(E) = \int \mu(E - t) d\lambda(t)$$

for every  $\mu$  and  $\lambda \in M$  and every Borel set  $E$ . Here

$$E - t = \{x - t: x \in E\}.$$

(e) Define  $\mu$  to be *discrete* if  $\mu$  is concentrated on a countable set; define  $\mu$  to be *continuous* if  $\mu(\{x\}) = 0$  for every point  $x \in R^1$ ; let  $m$  be Lebesgue measure on  $R^1$  (note that  $m \notin M$ ). Prove that  $\mu * \lambda$  is discrete if both  $\mu$  and  $\lambda$  are discrete, that  $\mu * \lambda$  is continuous if  $\mu$  is continuous and  $\lambda \in M$ , and that  $\mu * \lambda \ll m$  if  $\mu \ll m$ .

(f) Assume  $d\mu = f dm$ ,  $d\lambda = g dm$ ,  $f \in L^1(R^1)$ , and  $g \in L^1(R^1)$ , and prove that  $d(\mu * \lambda) = (f * g) dm$ .

(g) Properties (a) and (c) show that the Banach space  $M$  is what one calls a *commutative Banach algebra*. Show that (e) and (f) imply that the set of all discrete measures in  $M$  is a subalgebra of  $M$ , that the continuous measures form an ideal in  $M$ , and that the absolutely continuous measures (relative to  $m$ ) form an ideal in  $M$  which is isomorphic (as an algebra) to  $L^1(R^1)$ .

(h) Show that  $M$  has a unit, i.e., show that there exists a  $\delta \in M$  such that  $\delta * \mu = \mu$  for all  $\mu \in M$ .

(i) Only two properties of  $R^1$  have been used in this discussion:  $R^1$  is a commutative group (under addition), and there exists a translation invariant Borel measure  $m$  on  $R^1$  which is not identically 0 and which is finite on all compact subsets of  $R^1$ . Show that the same results hold if  $R^1$  is replaced by  $R^k$  or by  $T$  (the unit circle) or by  $T^k$  (the  $k$ -dimensional torus, the cartesian product of  $k$  copies of  $T$ ), as soon as the definitions are properly formulated.

6 (Polar coordinates in  $R^k$ .) Let  $S_{k-1}$  be the unit sphere in  $R^k$ , i.e., the set of all  $u \in R^k$  whose distance from the origin 0 is 1. Show that every  $x \in R^k$ , except for  $x = 0$ , has a unique representation of the form  $x = ru$ , where  $r$  is a positive real number and  $u \in S_{k-1}$ . Thus  $R^k - \{0\}$  may be regarded as the cartesian product  $(0, \infty) \times S_{k-1}$ .

Let  $m_k$  be Lebesgue measure on  $R^k$ , and define a measure  $\sigma_{k-1}$  on  $S_{k-1}$  as follows: If  $A \subset S_{k-1}$  and  $A$  is a Borel set, let  $\tilde{A}$  be the set of all points  $ru$ , where  $0 < r < 1$  and  $u \in A$ , and define

$$\sigma_{k-1}(A) = k \cdot m_k(\tilde{A}).$$



Prove that the formula

$$\int_{R^k} f \, dm_k = \int_0^\infty r^{k-1} \, dr \int_{S_{k-1}} f(ru) \, d\sigma_{k-1}(u)$$

is valid for every nonnegative Borel function  $f$  on  $R^k$ . Check that this coincides with familiar results when  $k = 2$  and when  $k = 3$ .

*Suggestion:* If  $0 < r_1 < r_2$  and if  $A$  is an open subset of  $S_{k-1}$ , let  $E$  be the set of all  $ru$  with  $r_1 < r < r_2$ ,  $u \in A$ , and verify that the formula holds for the characteristic function of  $E$ . Pass from these to characteristic functions of Borel sets in  $R^k$ .

7 Suppose  $(X, \mathcal{S}, \mu)$  and  $(Y, \mathcal{T}, \lambda)$  are  $\sigma$ -finite measure spaces, and suppose  $\psi$  is a measure on  $\mathcal{S} \times \mathcal{T}$  such that

$$\psi(A \times B) = \mu(A)\lambda(B)$$

whenever  $A \in \mathcal{S}$  and  $B \in \mathcal{T}$ . Prove that then  $\psi(E) = (\mu \times \lambda)(E)$  for every  $E \in \mathcal{S} \times \mathcal{T}$ .

8 (a) Suppose  $f$  is a real function on  $R^2$  such that each section  $f_x$  is Borel measurable and each section  $f^y$  is continuous. Prove that  $f$  is Borel measurable on  $R^2$ . Note the contrast between this and Example 8.9(c).

(b) Suppose  $g$  is a real function on  $R^k$  which is continuous in each of the  $k$  variables separately. More explicitly, for every choice of  $x_2, \dots, x_k$ , the mapping  $x_1 \rightarrow g(x_1, x_2, \dots, x_k)$  is continuous, etc. Prove that  $g$  is a Borel function.

*Hint:* If  $(i-1)/n = a_{i-1} \leq x \leq a_i = i/n$ , put

$$f_n(x, y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y).$$

9 Suppose  $E$  is a dense set in  $R^1$  and  $f$  is a real function on  $R^2$  such that (a)  $f_x$  is Lebesgue measurable for each  $x \in E$  and (b)  $f^y$  is continuous for almost all  $y \in R^1$ . Prove that  $f$  is Lebesgue measurable on  $R^2$ .

10 Suppose  $f$  is a real function on  $R^2$ ,  $f_x$  is Lebesgue measurable for each  $x$ , and  $f^y$  is continuous for each  $y$ . Suppose  $g: R^1 \rightarrow R^1$  is Lebesgue measurable, and put  $h(y) = f(g(y), y)$ . Prove that  $h$  is Lebesgue measurable on  $R^1$ .

*Hint:* Define  $f_n$  as in Exercise 8, put  $h_n(y) = f_n(g(y), y)$ , show that each  $h_n$  is measurable, and that  $h_n(y) \rightarrow h(y)$ .

11 Let  $\mathcal{B}_k$  be the  $\sigma$ -algebra of all Borel sets in  $R^k$ . Prove that  $\mathcal{B}_{m+n} = \mathcal{B}_m \times \mathcal{B}_n$ . This is relevant in Theorem 8.14.

12 Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt} \, dt \quad (x > 0)$$

to prove that

$$\lim_{A \rightarrow \infty} \int_0^A \frac{\sin x}{x} \, dx = \frac{\pi}{2}.$$

13. If  $\mu$  is a complex measure on a  $\sigma$ -algebra  $\mathfrak{M}$ , show that every set  $E \in \mathfrak{M}$  has a subset  $A$  for which

$$|\mu(A)| \geq \frac{1}{\pi} |\mu|(E).$$

*Suggestion:* There is a measurable real function  $\theta$  so that  $d\mu = e^{i\theta} d|\mu|$ . Let  $A_\alpha$  be the subset of  $E$  where  $\cos(\theta - \alpha) > 0$ , show that

$$\operatorname{Re} [e^{-i\alpha} \mu(A_\alpha)] = \int_{A_\alpha} \cos^+(\theta - \alpha) d|\mu|,$$

and integrate with respect to  $\alpha$  (as in Lemma 6.3).

Show, by an example, that  $1/\pi$  is the best constant in this inequality.

**14** Complete the following proof of Hardy's inequality (Chap. 3, Exercise 14): Suppose  $f \geq 0$  on  $(0, \infty)$ ,  $f \in L^p$ ,  $1 < p < \infty$ , and

$$F(x) = \frac{1}{x} \int_0^x f(t) dt.$$

Write  $xF(x) = \int_0^x f(t)t^\alpha t^{-\alpha} dt$ , where  $0 < \alpha < 1/q$ , use Hölder's inequality to get an upper bound for  $F(x)^p$ , and integrate to obtain

$$\int_0^\infty F^p(x) dx \leq (1 - \alpha q)^{1-p} (\alpha p)^{-1} \int_0^\infty f^p(t) dt.$$

Show that the best choice of  $\alpha$  yields

$$\int_0^\infty F^p(x) dx \leq \left( \frac{p}{p-1} \right)^p \int_0^\infty f^p(t) dt.$$

**15** Put  $\varphi(t) = 1 - \cos t$  if  $0 \leq t \leq 2\pi$ ,  $\varphi(t) = 0$  for all other real  $t$ . For  $-\infty < x < \infty$ , define

$$f(x) = 1, \quad g(x) = \varphi'(x), \quad h(x) = \int_{-\infty}^x \varphi(t) dt.$$

Verify the following statements about convolutions of these functions:

- (i)  $(f * g)(x) = 0$  for all  $x$ .
- (ii)  $(g * h)(x) = (\varphi * \varphi)(x) > 0$  on  $(0, 4\pi)$ .
- (iii) Therefore  $(f * g) * h = 0$ , whereas  $f * (g * h)$  is a positive constant.

But convolution is supposedly associative, by Fubini's theorem (Exercise 5(c)). What went wrong?

**16** Prove the following analogue of Minkowski's inequality, for  $f \geq 0$ :

$$\left\{ \int \left[ \int f(x, y) d\lambda(y) \right]^p d\mu(x) \right\}^{1/p} \leq \int \left[ \int f^p(x, y) d\mu(x) \right]^{1/p} d\lambda(y).$$

Supply the required hypotheses. (Many further developments of this theme may be found in [9].)