CHAPTER 2

Brownian Motion

2.1. Introduction

Brownian movement is the name given to the irregular movement of pollen, suspended in water, observed by the botanist Robert Brown in 1828. This random movement, now attributed to the buffeting of the pollen by water molecules, results in a dispersal or *diffusion* of the pollen in the water. The range of application of Brownian motion as defined here goes far beyond a study of microscopic particles in suspension and includes modeling of stock prices, of thermal noise in electrical circuits, of certain limiting behavior in queueing and inventory systems, and of random perturbations in a variety of other physical, biological, economic, and management systems. Furthermore, integration with respect to Brownian motion, developed in Chapter 3, gives us a unifying representation for a large class of martingales and diffusion processes. Diffusion processes represented this way exhibit a rich connection with the theory of partial differential equations (Chapter 4 and Section 5.7). In particular, to each such process there corresponds a second-order parabolic equation which governs the transition probabilities of the process.

The history of Brownian motion is discussed more extensively in Section 11; see also Chapters 2–4 in Nelson (1967).

1.1 Definition. A (*standard, one-dimensional*) *Brownian motion* is a continuous, adapted process $B = \{B_t, \mathscr{F}_t; 0 \le t < \infty\}$, defined on some probability space $(\Omega, \mathscr{F}, P)$, with the properties that $B_0 = 0$ a.s. and for $0 \le s < t$, the increment $B_t - B_s$ is independent of $\mathscr{F}_s$ and is normally distributed with mean zero and variance $t - s$. We shall speak sometimes of a Brownian motion $B = \{B_t, \mathscr{F}_t; 0 \le t \le T\}$ on $[0, T]$, for some $T > 0$, and the meaning of this terminology is apparent.

If B is a Brownian motion and $0 = t_0 < t_1 < \cdots < t_n < \infty$, then the increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^n$ are independent and the distribution of $B_{t_j} - B_{t_{j-1}}$ depends on t_j and t_{j-1} only through the difference $t_j - t_{j-1}$; to wit, it is normal with mean zero and variance $t_j - t_{j-1}$. We say that the process B has *stationary, independent increments*. It is easily verified that B is a square-integrable martingale and $\langle B\rangle_t = t, t \geq 0$.

The *filtration* $\{\mathscr{F}_t\}$ is a part of the definition of Brownian motion. However, if we are given $\{B_t; 0 \leq t < \infty\}$ but no filtration, and if we know that B has stationary, independent increments and that $B_t = B_t - B_0$ is normal with mean zero and variance t, then $\{B_t, \mathscr{F}_t^B; 0 \leq t < \infty\}$ is a Brownian motion (Problem 1.4). Moreover, if $\{\mathscr{F}_t\}$ is a "larger" filtration in the sense that $\mathscr{F}_t^B \subseteq \mathscr{F}_t$ for $t \geq 0$, and if $B_t - B_s$ is independent of $\mathscr{F}_s$ whenever $0 \leq s < t$, then $\{B_t, \mathscr{F}_t; 0 \leq t < \infty\}$ is also a Brownian motion.

It is often interesting, and necessary, to work with a filtration $\{\mathscr{F}_t\}$ which is larger than $\{\mathscr{F}_t^B\}$. For instance, we shall see in Example 5.3.5 that the stochastic differential equation (5.3.1) does not have a solution, unless we take the driving process W to be a Brownian motion with respect to a filtration which is *strictly larger* than $\{\mathscr{F}_t^W\}$. The desire to have existence of solutions to stochastic differential equations is a major motivation for allowing $\{\mathscr{F}_t\}$ in Definition 1.1 to be strictly larger than $\{\mathscr{F}_t^B\}$.

The first problem one encounters with Brownian motion is its *existence*. One approach to this question is to write down what the finite-dimensional distributions of this process (based on the stationarity, independence, and normality of its increments) must be, and then construct a probability measure and a process on an appropriate measurable space in such a way that we obtain the prescribed finite-dimensional distributions. This direct approach is the one most often used to construct a Markov process, but is rather lengthy and technical; we spell it out in Section 2. A more elegant approach for Brownian motion, which exploits the *Gaussian* property of this process, is based on Hilbert space theory and appears in Section 3; it is close in spirit to Wiener's (1923) original construction, which was modified by Lévy (1948) and later further simplified by Ciesielski (1961). Nothing in the remainder of the book depends on Section 3; however, Theorems 2.2 and 2.8 as well as Problem 2.9 will be useful in later developments.

Section 4 provides yet another proof for the existence of Brownian motion, this time based on the idea of the weak limit of a sequence of random walks. The properties of the space $C[0, \infty)$ developed in this section will be used extensively throughout the book.

Section 5 defines the *Markov property*, which is enjoyed by Brownian motion. Section 6 presents the *strong Markov property*, and, using a proof based on the optional sampling theorem for martingales, shows that Brownian motion is a strong Markov process. In Section 7 we discuss various choices of the filtration for Brownian motion. The central idea here is augmentation of the filtration generated by the process, in order to obtain a right-continuous filtration. Developing this material in the context of strong Markov processes requires no additional effort, and we adopt this level of generality.

Sections 8 and 9 are devoted to properties of Brownian motion. In Section 8 we compute distributions of a number of elementary Brownian functionals: among these are first passage times, last exit times, and time and level of the maximum over a fixed time-interval. Section 9 deals with almost sure properties of the Brownian sample path. Here we discuss its growth as $t \to \infty$, its oscillations near $t = 0$ (law of the iterated logarithm), its nowhere differentiability and nowhere monotonicity, and the topological perfectness of the set of times when the sample path is at the origin.

We conclude this introductory section with the Dynkin system theorem (Ash (1972), p. 169). This result will be used frequently in the sequel whenever we need to establish that a certain property, known to hold for a collection of sets closed under intersection, also holds for the σ-field generated by this collection. Our first application of this result occurs in Problem 1.4.

1.2 Definition. A collection $\mathscr{D}$ of subsets of a set Ω is called a *Dynkin system* if

(i) $\Omega \in \mathscr{D}$,
(ii) $A, B \in \mathscr{D}$ and $B \subseteq A$ imply $A \setminus B \in \mathscr{D}$,
(iii) $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{D}$ and $A_1 \subseteq A_2 \subseteq \cdots$ imply $\bigcup_{n=1}^{\infty} A_n \in \mathscr{D}$.

1.3 Dynkin System Theorem. *Let $\mathscr{C}$ be a collection of subsets of Ω which is closed under pairwise intersection. If $\mathscr{D}$ is a Dynkin system containing $\mathscr{C}$, then $\mathscr{D}$ also contains the σ-field $\sigma(\mathscr{C})$ generated by $\mathscr{C}$.*

1.4 Problem. Let $X = \{X_t; 0 \le t < \infty\}$ be a stochastic process for which $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent random variables, for every integer $n \ge 1$ and indices $0 = t_0 < t_1 < \cdots < t_n < \infty$. Then for any fixed $0 \le s < t < \infty$, the increment $X_t - X_s$ is independent of $\mathscr{F}_s^X$.

2.2. First Construction of Brownian Motion

A. The Consistency Theorem

Let $\mathbb{R}^{[0,\infty)}$ denote the set of all real-valued functions on $[0, \infty)$. An *n-dimensional cylinder set* in $\mathbb{R}^{[0,\infty)}$ is a set of the form

$$C \triangleq \{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_n)) \in A\}, \tag{2.1}$$

where $t_i \in [0, \infty)$, $i = 1, \ldots, n$, and $A \in \mathscr{B}(\mathbb{R}^n)$. Let $\mathscr{C}$ denote the field of all cylinder sets (of all finite dimensions) in $\mathbb{R}^{[0,\infty)}$, and let $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ denote the smallest σ-field containing $\mathscr{C}$.

2.1 Definition. Let T be the set of finite sequences $\underline{t} = (t_1, \ldots, t_n)$ of distinct, nonnegative numbers, where the length n of these sequences ranges over the set of positive integers. Suppose that for each $\underline{t}$ of length n, we have a probability measure $Q_{\underline{t}}$ on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. Then the collection $\{Q_{\underline{t}}\}_{\underline{t}\in T}$ is called a *family of finite-dimensional distributions*. This family is said to be *consistent* provided that the following two conditions are satisfied:

(a) if $\underline{s} = (t_{i_1}, t_{i_2}, \ldots, t_{i_n})$ is a permutation of $\underline{t} = (t_1, t_2, \ldots, t_n)$, then for any $A_i \in \mathscr{B}(\mathbb{R})$, $i = 1, \ldots, n$, we have
$$Q_{\underline{t}}(A_1 \times A_2 \times \cdots \times A_n) = Q_{\underline{s}}(A_{i_1} \times A_{i_2} \times \cdots \times A_{i_n});$$

(b) if $\underline{t} = (t_1, t_2, \ldots, t_n)$ with $n \geq 1$, $\underline{s} = (t_1, t_2, \ldots, t_{n-1})$, and $A \in \mathscr{B}(\mathbb{R}^{n-1})$, then
$$Q_{\underline{t}}(A \times \mathbb{R}) = Q_{\underline{s}}(A).$$

If we have a probability measure P on $(\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$, then we can define a family of finite-dimensional distributions by

(2.2) $$Q_{\underline{t}}(A) = P[\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_n)) \in A],$$

where $A \in \mathscr{B}(\mathbb{R}^n)$ and $\underline{t} = (t_1, \ldots, t_n) \in T$. This family is easily seen to be consistent. We are interested in the converse of this fact, because it will enable us to construct a probability measure P from the finite-dimensional distributions of Brownian motion.

2.2 Theorem (Daniell (1918), Kolmogorov (1933)). *Let $\{Q_{\underline{t}}\}$ be a consistent family of finite-dimensional distributions. Then there is a probability measure P on $(\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$, such that* (2.2) *holds for every $\underline{t} \in T$.*

Proof. We begin by defining a set function Q on the field of cylinders $\mathscr{C}$. If C is given by (2.1) and $\underline{t} = (t_1, t_2, \ldots, t_n) \in T$, we set

(2.3) $$Q(C) = Q_{\underline{t}}(A), \quad C \in \mathscr{C}.$$

2.3 Problem. The set function Q is well defined and finitely additive on $\mathscr{C}$, with $Q(\mathbb{R}^{[0,\infty)}) = 1$.

We now prove the countable additivity of Q on $\mathscr{C}$, and we can then draw on the Carathéodory extension theorem to assert the existence of the desired extension P of Q to $\mathscr{B}(\mathbb{R}^{[0,\infty)})$. Thus, suppose $\{B_k\}_{k=1}^{\infty}$ is a sequence of disjoint sets in $\mathscr{C}$ with $B \triangleq \bigcup_{k=1}^{\infty} B_k$ also in $\mathscr{C}$. Let $C_m = B \setminus \bigcup_{k=1}^{m} B_k$, so
$$Q(B) = Q(C_m) + \sum_{k=1}^{m} Q(B_k).$$
Countable additivity will follow from

(2.4) $$\lim_{m\to\infty} Q(C_m) = 0.$$

Now $Q(C_m) = Q(C_{m+1}) + Q(B_{m+1}) \geq Q(C_{m+1})$, so the limit in (2.4) exists. Assume that this limit is equal to $\varepsilon > 0$, and note that $\bigcap_{m=1}^{\infty} C_m = \varnothing$.

From $\{C_m\}_{m=1}^{\infty}$ we may construct another sequence $\{D_m\}_{m=1}^{\infty}$ which has the properties $D_1 \supseteq D_2 \supseteq \cdots$, $\bigcap_{m=1}^{\infty} D_m = \emptyset$, and $\lim_{m\to\infty} Q(D_m) = \varepsilon > 0$. Furthermore, each D_m has the form

$$D_m = \{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_m)) \in A_m\}$$

for some $A_m \in \mathscr{B}(\mathbb{R}^m)$, and the finite sequence $\underline{t}_m \triangleq (t_1, \ldots, t_m) \in T$ is an extension of the finite sequence $\underline{t}_{m-1} \triangleq (t_1, \ldots, t_{m-1}) \in T$, $m \geq 2$. This may be accomplished as follows. Each C_k has a form

$$C_k = \{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_{m_k})) \in A_{m_k}\}; \quad A_{m_k} \in \mathscr{B}(\mathbb{R}^{m_k}),$$

where $\underline{t}_{m_k} = (t_1, \ldots, t_{m_k}) \in T$. Since $C_{k+1} \subseteq C_k$, we can choose these representations so that $\underline{t}_{m_{k+1}}$ is an extension of $\underline{t}_{m_k}$ and $A_{m_{k+1}} \subseteq A_{m_k} \times \mathbb{R}^{m_{k+1}-m_k}$. Define

$$D_1 = \{\omega; \omega(t_1) \in \mathbb{R}\}, \ldots, D_{m_1-1} = \{\omega; (\omega(t_1), \ldots, \omega(t_{m_1-1})) \in \mathbb{R}^{m_1-1}\}$$

and $D_{m_1} = C_1$, as well as

$$D_{m_1+1} = \{\omega; (\omega(t_1), \ldots, \omega(t_{m_1}), \omega(t_{m_1+1})) \in A_{m_1} \times \mathbb{R}\}, \ldots,$$

$$D_{m_2-1} = \{\omega; (\omega(t_1), \ldots, \omega(t_{m_1}), \omega(t_{m_1+1}), \ldots, \omega(t_{m_2-1})) \in A_{m_1} \times \mathbb{R}^{m_2-m_1-1}\}$$

and $D_{m_2} = C_2$. Continue this process, and note that by construction $\bigcap_{m=1}^{\infty} D_m = \bigcap_{m=1}^{\infty} C_m = \emptyset$.

2.4 Problem. Let Q be a probability measure on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$. We say that $A \in \mathscr{B}(\mathbb{R}^n)$ is *regular* if for every probability measure Q on $(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n))$ and for every $\varepsilon > 0$, there is a closed set F and an open set G such that $F \subseteq A \subseteq G$ and $Q(G \backslash F) < \varepsilon$. Show that every set in $\mathscr{B}(\mathbb{R}^n)$ is regular. (*Hint*: Show that the collection of regular sets is a σ-field containing all closed sets.)

According to Problem 2.4, there exists for each m a closed set $F_m \subseteq A_m$ such that $Q_{\underline{t}_m}(A_m \backslash F_m) < \varepsilon/2^m$. By intersecting F_m with a sufficiently large closed sphere centered at the origin, we obtain a compact set K_m such that, with

$$E_m \triangleq \{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_m)) \in K_m\},$$

we have $E_m \subseteq D_m$ and

$$Q(D_m \backslash E_m) = Q_{\underline{t}_m}(A_m \backslash K_m) < \frac{\varepsilon}{2^m}.$$

The sequence $\{E_m\}$ may fail to be nonincreasing, so we define

$$\tilde{E}_m = \bigcap_{k=1}^{m} E_k,$$

and we have

$$\tilde{E}_m = \{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \ldots, \omega(t_m)) \in \tilde{K}_m\},$$

where

$$\tilde{K}_m = (K_1 \times \mathbb{R}^{m-1}) \cap (K_2 \times \mathbb{R}^{m-2}) \cap \cdots \cap (K_{m-1} \times \mathbb{R}) \cap K_m,$$

which is compact. We can bound $Q_{\underline{t}_m}(\tilde{K}_m)$ away from zero, since

$$\begin{aligned} Q_{\underline{t}_m}(\tilde{K}_m) &= Q(\tilde{E}_m) = Q(D_m) - Q(D_m \backslash \tilde{E}_m) \\ &= Q(D_m) - Q\left(\bigcup_{k=1}^{m} (D_m \backslash E_k)\right) \\ &\geq Q(D_m) - Q\left(\bigcup_{k=1}^{m} (D_k \backslash E_k)\right) \\ &\geq \varepsilon - \sum_{k=1}^{m} \frac{\varepsilon}{2^k} > 0. \end{aligned}$$

Therefore, $\tilde{K}_m$ is nonempty for each m, and we can choose $(x_1^{(m)}, \ldots, x_m^{(m)}) \in \tilde{K}_m$. Being contained in the compact set $\tilde{K}_1$, the sequence $\{x_1^{(m)}\}_{m=1}^{\infty}$ must have a convergent subsequence $\{x_1^{(m_k)}\}_{k=1}^{\infty}$ with limit x_1. But $\{x_1^{(m_k)}, x_2^{(m_k)}\}_{k=2}^{\infty}$ is contained in $\tilde{K}_2$, so it has a convergent subsequence with limit (x_1, x_2). Continuing this process, we can construct $(x_1, x_2, \ldots) \in \mathbb{R} \times \mathbb{R} \times \cdots$, such that $(x_1, \ldots, x_m) \in \tilde{K}_m$ for each m. Consequently, the set

$$S = \{\omega \in \mathbb{R}^{[0,\infty)};\ \omega(t_i) = x_i,\ i = 1, 2, \ldots\}$$

is contained in each $\tilde{E}_m$, and hence in each D_m. This contradicts the fact that $\bigcap_{m=1}^{\infty} D_m = \varnothing$. We conclude that (2.4) holds. □

Our aim is to construct a probability measure P on $(\Omega, \mathscr{F}) \triangleq (\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$ so that the process $B = \{B_t, \mathscr{F}_t^B;\ 0 \leq t < \infty\}$ defined by $B_t(\omega) \triangleq \omega(t)$, the *coordinate mapping process*, is almost a standard, one-dimensional Brownian motion under P. We say "almost" because we leave aside the requirement of sample path continuity for the moment and concentrate on the finite-dimensional distributions. Recalling the discussion following Definition 1.1, we see that whenever $0 = s_0 < s_1 < s_2 < \cdots < s_n$, the cumulative distribution function for $(B_{s_1}, \ldots, B_{s_n})$ must be

(2.5)
$$\begin{aligned} &F_{(s_1,\ldots,s_n)}(x_1, \ldots, x_n) \\ &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_n} p(s_1; 0, y_1) p(s_2 - s_1; y_1, y_2) \ldots \\ &\qquad \ldots p(s_n - s_{n-1}; y_{n-1}, y_n)\, dy_n \ldots dy_2\, dy_1 \end{aligned}$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$, where p is the Gaussian kernel

(2.6)
$$p(t; x, y) \triangleq \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}, \quad t > 0,\ x,\ y \in \mathbb{R}.$$

The reader can verify (and should, if he has never done so!) that (2.5) is equivalent to the statement that the increments $\{B_{s_j} - B_{s_{j-1}}\}_{j=1}^{n}$ are independent, and $B_{s_j} - B_{s_{j-1}}$ is normally distributed with mean zero and variance $s_j - s_{j-1}$.

Now let $\underline{t} = (t_1, t_2, \ldots, t_n)$, where the t_j are not necessarily ordered but are distinct. Let the random vector $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ have the distribution determined by (2.5) (where the t_j must be ordered from smallest to largest to obtain $(s_1, \ldots, s_n)$ appearing in (2.5)). For $A \in \mathscr{B}(\mathbb{R}^n)$, let $Q_{\mathbf{t}}(A)$ be the probability under this distribution that $(B_{t_1}, B_{t_2}, \ldots, B_{t_n})$ is in A. This defines a family of finite-dimensional distributions $\{Q_{\mathbf{t}}\}_{\mathbf{t} \in \mathscr{T}}$.

2.5 Problem. Show that the just defined family $\{Q_{\mathbf{t}}\}_{\mathbf{t} \in \mathscr{T}}$ is consistent.

2.6 Corollary to Theorem 2.2. *There is a probability measure P on $(\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$, under which the coordinate mapping process*

$$B_t(\omega) = \omega(t); \quad \omega \in \mathbb{R}^{[0,\infty)}, \ t \geq 0,$$

has stationary, independent increments. An increment $B_t - B_s$, where $0 \leq s < t$, is normally distributed with mean zero and variance $t - s$.

B. The Kolmogorov-Čentsov Theorem

Our construction of Brownian motion would now be complete, were it not for the fact that we have built the process on the sample space $\mathbb{R}^{[0,\infty)}$ of all real-valued functions on $[0, \infty)$ rather than on the space $C[0, \infty)$ of continuous functions on this half-line. One might hope to overcome this difficulty by showing that the probability measure P in Corollary 2.6 assigns measure one to $C[0, \infty)$. However, as the next problem shows, $C[0, \infty)$ is not in the σ-field $\mathscr{B}(\mathbb{R}^{[0,\infty)})$, so $P(C[0, \infty))$ is not defined. This failure is a manifestation of the fact that the σ-field $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ is, quite uncomfortably, "too small" for a space as big as $\mathbb{R}^{[0,\infty)}$; no set in $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ can have restrictions on uncountably many coordinates. In contrast to the space $C[0, \infty)$, it is not possible to determine a function in $\mathbb{R}^{[0,\infty)}$ by specifying its values at only countably many coordinates. Consequently, the next theorem takes a different approach, which is to construct a continuous modification of the coordinate mapping process in Corollary 2.6.

2.7 Exercise. Show that the only $\mathscr{B}(\mathbb{R}^{[0,\infty)})$-measurable set contained in $C[0, \infty)$ is the empty set. (*Hint:* A typical set in $\mathscr{B}(\mathbb{R}^{[0,\infty)})$ has the form

$$E = \{\omega \in \mathbb{R}^{[0,\infty)}; (\omega(t_1), \omega(t_2), \ldots) \in A\},$$

where $A \in \mathscr{B}(\mathbb{R} \times \mathbb{R} \times \cdots)$).

2.8 Theorem (Kolmogorov, Čentsov (1956a)). *Suppose that a process $X = \{X_t; 0 \leq t \leq T\}$ on a probability space $(\Omega, \mathscr{F}, P)$ satisfies the condition*

$$E|X_t - X_s|^\alpha \leq C|t - s|^{1+\beta}, \quad 0 \leq s, t \leq T, \tag{2.7}$$

for some positive constants α, β, and C. Then there exists a continuous modification $\tilde{X} = \{\tilde{X}_t; 0 \leq t \leq T\}$ of X, which is locally Hölder-continuous with exponent γ

for every $\gamma \in (0, \beta/\alpha)$, *i.e.*,

$$(2.8) \qquad P\left[\omega;\ \sup_{\substack{0<t-s<h(\omega)\\ s,t\in[0,T]}} \frac{|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)|}{|t-s|^\gamma} \le \delta\right] = 1,$$

where $h(\omega)$ *is an a.s. positive random variable and* $\delta > 0$ *is an appropriate constant.*

Proof. For notational simplicity, we take $T = 1$. Much of what follows is a consequence of the Čebyšev inequality. First, for any $\varepsilon > 0$, we have

$$P[|X_t - X_s| \ge \varepsilon] \le \frac{E|X_t - X_s|^\alpha}{\varepsilon^\alpha} \le C\varepsilon^{-\alpha}|t-s|^{1+\beta},$$

and so $X_s \to X_t$ in probability as $s \to t$. Second, setting $t = k/2^n$, $s = (k-1)/2^n$, and $\varepsilon = 2^{-\gamma n}$ (where $0 < \gamma < \beta/\alpha$) in the preceding inequality, we obtain

$$P[|X_{k/2^n} - X_{(k-1)/2^n}| \ge 2^{-\gamma n}] \le C2^{-n(1+\beta-\alpha\gamma)},$$

and consequently,

$$(2.9) \qquad P\left[\max_{1\le k\le 2^n} |X_{k/2^n} - X_{(k-1)/2^n}| \ge 2^{-\gamma n}\right]$$

$$\le P\left[\bigcup_{k=1}^{2^n} |X_{k/2^n} - X_{(k-1)/2^n}| \ge 2^{-\gamma n}\right]$$

$$\le C2^{-n(\beta-\alpha\gamma)}.$$

The last expression is the general term of a convergent series; by the Borel-Cantelli lemma, there is a set $\Omega^* \in \mathscr{F}$ with $P(\Omega^*) = 1$ such that for each $\omega \in \Omega^*$,

$$(2.10) \qquad \max_{1\le k\le 2^n} |X_{k/2^n}(\omega) - X_{(k-1)/2^n}(\omega)| < 2^{-\gamma n}, \quad \forall\, n \ge n^*(\omega),$$

where $n^*(\omega)$ is a positive, integer-valued random variable.

For each integer $n \ge 1$, let us consider the partition $D_n = \{(k/2^n); k = 0, 1, \ldots, 2^n\}$ of $[0,1]$, and let $D = \bigcup_{n=1}^\infty D_n$ be the set of dyadic rationals in $[0,1]$. We shall fix $\omega \in \Omega^*$, $n \ge n^*(\omega)$, and show that for every $m > n$, we have

$$(2.11) \qquad |X_t(\omega) - X_s(\omega)| \le 2\sum_{j=n+1}^{m} 2^{-\gamma j}; \quad \forall\, t, s \in D_m, 0 < t - s < 2^{-n}.$$

For $m = n + 1$, we can only have $t = (k/2^m)$, $s = ((k-1)/2^m)$, and (2.11) follows from (2.10). Suppose (2.11) is valid for $m = n+1, \ldots, M-1$. Take $s < t$, $s, t \in D_M$, consider the numbers $t^1 = \max\{u \in D_{M-1}; u \le t\}$ and $s^1 = \min\{u \in D_{M-1}; u \ge s\}$, and notice the relationships $s \le s^1 \le t^1 \le t$, $s^1 - s \le 2^{-M}$, $t - t^1 \le 2^{-M}$. From (2.10) we have $|X_{s^1}(\omega) - X_s(\omega)| \le 2^{-\gamma M}$, $|X_t(\omega) - X_{t^1}(\omega)| \le 2^{-\gamma M}$, and from (2.11) with $m = M - 1$,

$$|X_{t^1}(\omega) - X_{s^1}(\omega)| \le 2\sum_{j=n+1}^{M-1} 2^{-\gamma j}.$$

We obtain (2.11) for $m = M$.

We can show now that $\{X_t(\omega); t \in D\}$ is uniformly continuous in t for every $\omega \in \Omega^*$. For any numbers $s, t \in D$ with $0 < t - s < h(\omega) \triangleq 2^{-n^*(\omega)}$, we select $n \geq n^*(\omega)$ such that $2^{-(n+1)} \leq t - s < 2^{-n}$. We have from (2.11)

$$(2.12) \quad |X_t(\omega) - X_s(\omega)| \leq 2 \sum_{j=n+1}^{\infty} 2^{-\gamma j} \leq \delta |t - s|^\gamma, \quad 0 < t - s < h(\omega),$$

where $\delta = 2/(1 - 2^{-\gamma})$. This proves the desired uniform continuity.

We define $\tilde{X}$ as follows. For $\omega \notin \Omega^*$, set $\tilde{X}_t(\omega) = 0$, $0 \leq t \leq 1$. For $\omega \in \Omega^*$ and $t \in D$, set $\tilde{X}_t(\omega) = X_t(\omega)$. For $\omega \in \Omega^*$ and $t \in [0, 1] \cap D^c$, choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$; uniform continuity and the Cauchy criterion imply that $\{X_{s_n}(\omega)\}_{n=1}^{\infty}$ has a limit which depends on t but not on the particular sequence $\{s_n\}_{n=1}^{\infty} \subseteq D$ chosen to converge to t, and we set $\tilde{X}_t(\omega) = \lim_{s_n \to t} X_{s_n}(\omega)$. The resulting process $\tilde{X}$ is thereby continuous; indeed, $\tilde{X}$ satisfies (2.12), so (2.8) is established.

To see that $\tilde{X}$ is a modification of X, observe that $\tilde{X}_t = X_t$ a.s. for $t \in D$; for $t \in [0, 1] \cap D^c$ and $\{s_n\}_{n=1}^{\infty} \subseteq D$ with $s_n \to t$, we have $X_{s_n} \to X_t$ in probability and $X_{s_n} \to \tilde{X}_t$ a.s., so $\tilde{X}_t = X_t$ a.s. □

2.9 Problem. A *random field* is a collection of random variables $\{X_t; t \in \mathscr{A}\}$, where $\mathscr{A}$ is a partially ordered set. Suppose $\{X_t; t \in [0, T]^d\}$, $d \geq 2$, is a random field satisfying

$$(2.13) \qquad E|X_t - X_s|^\alpha \leq C\|t - s\|^{d+\beta}$$

for some positive constants α, β, and C. Show that the conclusion of Theorem 2.8 holds, with (2.8) replaced by

$$(2.14) \qquad P\left[\omega; \sup_{\substack{0<\|t-s\|<h(\omega) \\ s,t \in [0,T]^d}} \frac{|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)|}{\|t - s\|^\gamma} \leq \delta\right] = 1.$$

2.10 Problem. Show that if $B_t - B_s$, $0 \leq s < t$, is normally distributed with mean zero and variance $t - s$, then for each positive integer n, there is a positive constant C_n for which

$$E|B_t - B_s|^{2n} = C_n|t - s|^n.$$

2.11 Corollary to Theorem 2.8. *There is a probability measure P on $(\mathbb{R}^{[0,\infty)}, \mathscr{B}(\mathbb{R}^{[0,\infty)}))$, and a stochastic process $W = \{W_t, \mathscr{F}_t^W; t \geq 0\}$ on the same space, such that under P, W is a Brownian motion.*

PROOF. According to Theorem 2.8 and Problem 2.10, there is for each $T > 0$ a modification W^T of the process B in Corollary 2.6 such that W^T is continuous on $[0, T]$. Let

$$\Omega_T = \{\omega; W_t^T(\omega) = B_t(\omega) \text{ for every rational } t \in [0, T]\},$$

so $P(\Omega_T) = 1$. On $\tilde{\Omega} \triangleq \bigcap_{T=1}^{\infty} \Omega_T$, we have for positive integers T_1 and T_2,

$$W_t^{T_1}(\omega) = W_t^{T_2}(\omega), \text{ for every rational } t \in [0, T_1 \wedge T_2].$$

Since both processes are continuous on $[0, T_1 \wedge T_2]$, we must have $W_t^{T_1}(\omega) = W_t^{T_2}(\omega)$ for every $t \in [0, T_1 \wedge T_2]$, $\omega \in \tilde{\Omega}$. Define $W_t(\omega)$ to be this common value. For $\omega \notin \tilde{\Omega}$, set $W_t(\omega) = 0$ for all $t \geq 0$. □

2.12 Remark. Actually, for P-a.e. $\omega \in \mathbb{R}^{[0,\infty)}$, the Brownian sample path $\{W_t(\omega); 0 \leq t < \infty\}$ is locally Hölder-continuous with exponent γ, for every $\gamma \in (0, 1/2)$. This is a consequence of Theorem 2.8 and Problem 2.10.

2.3. Second Construction of Brownian Motion

This section provides a succinct, self-contained construction of Brownian motion. It may be omitted without loss of continuity.

Let us suppose that $\{B_t, \mathscr{F}_t; t \geq 0\}$ is a Brownian motion, fix $0 \leq s < t < \infty$, and set $\theta \triangleq (t + s)/2$; then, conditioned on $B_s = x$ and $B_t = z$, the random variable B_θ is normal with mean $\mu \triangleq (x + z)/2$ and variance $\sigma^2 \triangleq (t - s)/4$. To verify this, observe that the known distribution and independence of the increments B_s, $B_\theta - B_s$, and $B_t - B_\theta$ lead to the joint density

$$P[B_s \in dx, B_\theta \in dy, B_t \in dz] = p(s; 0, x) p\left(\frac{t-s}{2}; x, y\right) p\left(\frac{t-s}{2}; y, z\right) dx\, dy\, dz$$

$$= p(s; 0, x) p(t - s; x, z) \cdot \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\} dx\, dy\, dz$$

in the notation of (2.6), after a bit of algebra. Dividing by

$$P[B_s \in dx, B_t \in dz] = p(s; 0, x) p(t - s; x, z)\, dx\, dz,$$

we obtain

$$P[B_{(t+s)/2} \in dy \mid B_s = x, B_t = z] = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/2\sigma^2}\, dy. \tag{3.1}$$

The simple form of this conditional distribution for $B_{(t+s)/2}$ suggests that we can construct Brownian motion on some finite time-interval, say $[0, 1]$, by interpolation. Once we have completed the construction on $[0, 1]$, a simple "patching together" of a sequence of such Brownian motions will result in a Brownian motion defined for all $t \geq 0$.

To carry out this program, we begin with a countable collection $\{\xi_k^{(n)}; k \in I(n), n = 0, 1, \ldots\}$ of independent, standard (zero mean and unit variance) normal random variables on a probability space $(\Omega, \mathscr{F}, P)$. Here $I(n)$ is the set of odd integers between 0 and 2^n; i.e., $I(0) = \{1\}$, $I(1) = \{1\}$, $I(2) = \{1, 3\}$, etc. For each $n \geq 0$, we define a process $B^{(n)} = \{B_t^{(n)}; 0 \leq t \leq 1\}$ by recursion and linear interpolation, as follows. For $n \geq 1$, $B_{k/2^{n-1}}^{(n)}$ will agree with $B_{k/2^{n-1}}^{(n-1)}$, $k = 0, 1, \ldots, 2^{n-1}$. Thus, for each value of n, we need only specify $B_{k/2^n}^{(n)}$ for $k \in I(n)$. We set

$$B_0^{(0)} = 0, \quad B_1^{(0)} = \xi_1^{(0)}.$$

If the values of $B_{k/2^{n-1}}^{(n-1)}$, $k = 0, 1, \ldots, 2^{n-1}$ have been specified (so $B_t^{(n-1)}$ is defined for $0 \le t \le 1$ by piecewise-linear interpolation) and $k \in I(n)$, we denote $s = (k-1)/2^n$, $t = (k+1)/2^n$, $\mu = \frac{1}{2}(B_s^{(n-1)} + B_t^{(n-1)})$, and $\sigma^2 = (t-s)/4 = 1/2^{n+1}$ and set, in accordance with (3.1),

$$B_{k/2^n}^{(n)} \equiv B_{(t+s)/2}^{(n)} \triangleq \mu + \sigma \xi_k^{(n)}.$$

We shall show that, almost surely, $B_t^{(n)}$ converges uniformly in t to a continuous function B_t, and $\{B_t, \mathscr{F}_t^B; 0 \le t \le 1\}$ is a Brownian motion.

Our first step is to give a more convenient representation for the processes $B^{(n)}$, $n = 0, 1, \ldots$. We define the *Haar functions* by $H_1^{(0)}(t) = 1$, $0 \le t \le 1$, and for $n \ge 1$, $k \in I(n)$,

$$H_k^{(n)}(t) = \begin{cases} 2^{(n-1)/2}, & \dfrac{k-1}{2^n} \le t < \dfrac{k}{2^n}, \\[2ex] -2^{(n-1)/2}, & \dfrac{k}{2^n} \le t < \dfrac{k+1}{2^n}, \\[2ex] 0, & \text{otherwise}. \end{cases}$$

We define the *Schauder functions* by

$$S_k^{(n)}(t) = \int_0^t H_k^{(n)}(u)\,du, \quad 0 \le t \le 1, n \ge 0, k \in I(n).$$

Note that $S_1^{(0)}(t) = t$, and for $n \ge 1$ the graphs of $S_k^{(n)}$ are little tents of height $2^{-(n+1)/2}$ centered at $k/2^n$ and nonoverlapping for different values of $k \in I(n)$. It is clear that $B_t^{(0)} = \xi_1^{(0)} S_1^{(0)}(t)$, and by induction on n, it is easily verified that

$$B_t^{(n)}(\omega) = \sum_{m=0}^{n} \sum_{k \in I(m)} \xi_k^{(m)}(\omega) S_k^{(m)}(t), \quad 0 \le t \le 1, n \ge 0. \tag{3.2}$$

3.1 Lemma. *As $n \to \infty$, the sequence of functions $\{B_t^{(n)}(\omega); 0 \le t \le 1\}$, $n \ge 0$, given by (3.2) converges uniformly in t to a continuous function $\{B_t(\omega); 0 \le t \le 1\}$, for a.e. $\omega \in \Omega$.*

PROOF. Define $b_n = \max_{k \in I(n)} |\xi_k^{(n)}|$. For $x > 0$

$$P[|\xi_k^{(n)}| > x] = \sqrt{\frac{2}{\pi}} \int_x^\infty e^{-u^2/2}\,du \le \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{u}{x} e^{-u^2/2}\,du = \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x}, \tag{3.3}$$

which gives

$$P[b_n > n] = P\left[\bigcup_{k \in I_n} \{|\xi_k^{(n)}| > n\}\right] \le 2^n P[|\xi_1^{(n)}| > n] \le \sqrt{\frac{2}{\pi}} \frac{2^n e^{-n^2/2}}{n}, \quad n \ge 1.$$

Now $\sum_{n=1}^{\infty} 2^n e^{-n^2/2}/n < \infty$, so the Borel-Cantelli lemma implies that there is a set $\tilde{\Omega}$ with $P(\tilde{\Omega}) = 1$ such that for each $\omega \in \tilde{\Omega}$ there is an integer $n(\omega)$ satisfying $b_n(\omega) \le n$ for all $n \ge n(\omega)$. But then

$$\sum_{n=n(\omega)}^{\infty} \sum_{k \in I(n)} |\xi_k^{(n)} S_k^{(n)}(t)| \le \sum_{n=n(\omega)}^{\infty} n 2^{-(n+1)/2} < \infty;$$

so for $\omega \in \tilde{\Omega}$, $B_t^{(n)}(\omega)$ converges uniformly in t to a limit $B_t(\omega)$. Continuity of $\{B_t(\omega); 0 \le t \le 1\}$ follows from the uniformity of the convergence. □

Under the inner product $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$, $L^2[0,1]$ is a Hilbert space, and the Haar functions $\{H_k^{(n)}; k \in I(n), n \ge 0\}$ form a complete, orthonormal system (see, e.g., Kaczmarz & Steinhaus (1951), but also Exercise 3.3 later). The Parseval equality

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \sum_{k \in I(n)} \langle f, H_k^{(n)} \rangle \langle g, H_k^{(n)} \rangle,$$

applied to $f = 1_{[0,t]}$ and $g = 1_{[0,s]}$ yields

$$\sum_{n=0}^{\infty} \sum_{k \in I(n)} S_k^{(n)}(t) S_k^{(n)}(s) = s \wedge t; \quad 0 \le s, t \le 1. \tag{3.4}$$

3.2 Theorem. *With* $\{B_t^{(n)}\}_{n=1}^{\infty}$ *defined by* (3.2) *and* $B_t = \lim_{n\to\infty} B_t^{(n)}$, *the process* $\{B_t, \mathscr{F}_t^B; 0 \le t \le 1\}$ *is a Brownian motion on* $[0,1]$.

PROOF. It suffices to prove that, for $0 = t_0 < t_1 < \cdots < t_n \le 1$, the increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^{n}$ are independent, normally distributed, with mean zero and variance $t_j - t_{j-1}$. For this, we show that for $\lambda_j \in \mathbb{R}, j = 1, \ldots, n$ and $i = \sqrt{-1}$,

$$E\left[\exp\left\{i \sum_{j=1}^{n} \lambda_j (B_{t_j} - B_{t_{j-1}})\right\}\right] = \prod_{j=1}^{n} \exp\left\{-\frac{1}{2}\lambda_j^2 (t_j - t_{j-1})\right\}. \tag{3.5}$$

Set $\lambda_{n+1} = 0$. Using the independence and standard normality of the random variables $\{\xi_k^{(n)}\}$, we have from (3.2)

$$\begin{aligned}
&E\left[\exp\left\{-i \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) B_{t_j}^{(M)}\right\}\right] \\
&= E\left[\exp\left\{-i \sum_{m=0}^{M} \sum_{k \in I(m)} \xi_k^{(m)} \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) S_k^{(m)}(t_j)\right\}\right] \\
&= \prod_{m=0}^{M} \prod_{k \in I(m)} E\left[\exp\left\{-i \xi_k^{(m)} \sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) S_k^{(m)}(t_j)\right\}\right] \\
&= \prod_{m=0}^{M} \prod_{k \in I(m)} \exp\left[-\frac{1}{2}\left\{\sum_{j=1}^{n} (\lambda_{j+1} - \lambda_j) S_k^{(m)}(t_j)\right\}^2\right] \\
&= \exp\left[-\frac{1}{2} \sum_{j=1}^{n} \sum_{l=1}^{n} (\lambda_{j+1} - \lambda_j)(\lambda_{l+1} - \lambda_l) \sum_{m=0}^{M} \sum_{k \in I(m)} S_k^{(m)}(t_l) S_k^{(m)}(t_j)\right].
\end{aligned}$$

Letting $M \to \infty$ and using (3.4), we obtain

$$
\begin{aligned}
E\left[\exp\left\{i\sum_{j=1}^{n}\lambda_j(B_{t_j}-B_{t_{j-1}})\right\}\right] &= E\left[\exp\left\{-i\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)B_{t_j}\right\}\right] \\
&= \exp\left\{-\sum_{j=1}^{n-1}\sum_{i=j+1}^{n}(\lambda_{j+1}-\lambda_j)(\lambda_{i+1}-\lambda_i)t_j - \frac{1}{2}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)^2 t_j\right\} \\
&= \exp\left\{-\sum_{j=1}^{n-1}(\lambda_{j+1}-\lambda_j)(-\lambda_{j+1})t_j - \frac{1}{2}\sum_{j=1}^{n}(\lambda_{j+1}-\lambda_j)^2 t_j\right\} \\
&= \exp\left\{\frac{1}{2}\sum_{j=1}^{n-1}(\lambda_{j+1}^2-\lambda_j^2)t_j - \frac{1}{2}\lambda_n^2 t_n\right\} \\
&= \prod_{j=1}^{n}\exp\left\{-\frac{1}{2}\lambda_j^2(t_j-t_{j-1})\right\}.
\end{aligned}
$$

□

3.3 Exercise. Prove Theorem 3.2 without resort to the Parseval identity (3.4), by completing the following steps.

(a) The increments $\{B^{(n)}_{k/2^n} - B^{(n)}_{(k-1)/2^n}\}_{k=1}^{2^n}$ are independent, normal random variables with mean zero and variance $1/2^n$.

(b) If $0 = t_0 < t_1 < \cdots < t_n \le 1$ and each t_j is a dyadic rational, then the increments $\{B_{t_j} - B_{t_{j-1}}\}_{j=1}^n$ are independent, normal random variables with mean zero and variance $(t_j - t_{j-1})$.

(c) The assertion in (b) holds even if $\{t_j\}_{j=1}^n$ is not contained in the set of dyadic rationals.

3.4 Corollary. *There is a probability space $(\Omega, \mathscr{F}, P)$ and a stochastic process $B = \{B_t, \mathscr{F}_t^B; 0 \le t < \infty\}$ on it, such that B is a standard, one-dimensional Brownian motion.*

Proof. According to Theorem 3.2, there is a sequence $(\Omega_n, \mathscr{F}_n, P_n)$, $n = 1, 2, \ldots$ of probability spaces together with a Brownian motion $\{X_t^{(n)}: 0 \le t \le 1\}$ on each space. Let $\Omega = \Omega_1 \times \Omega_2 \times \cdots$, $\mathscr{F} = \mathscr{F}_1 \otimes \mathscr{F}_2 \otimes \cdots$, and $P = P_1 \times P_2 \times \cdots$. Define B on Ω recursively by

$$B_t = X_t^{(1)}, \quad 0 \le t \le 1,$$

$$B_t = B_n + X_{t-n}^{(n+1)}, \quad n \le t \le n+1.$$

This process is clearly continuous, and the increments are easily seen to be independent and normal with zero mean and the proper variances. □

2.4. The Space $C[0, \infty)$, Weak Convergence, and the Wiener Measure

The sample spaces for the Brownian motions we built in Sections 2 and 3 were, respectively, the space $\mathbb{R}^{[0,\infty)}$ of all real-valued functions on $[0,\infty)$ and a space Ω rich enough to carry a countable collection of independent, standard normal random variables. The "canonical" space for Brownian motion, the one most convenient for many future developments, is $C[0,\infty)$, the space of all continuous, real-valued functions on $[0,\infty)$ with metric

$$\rho(\omega_1,\omega_2) \triangleq \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0\le t\le n} (|\omega_1(t)-\omega_2(t)| \wedge 1). \tag{4.1}$$

In this section, we show how to construct a measure, called *Wiener measure*, on this space so that the coordinate mapping process is Brownian motion. This construction is given as the proof of Theorem 4.20 (Donsker's invariance principle) and involves the notion of weak convergence of random walks to Brownian motion.

4.1 Problem. Show that ρ defined by (4.1) is a metric on $C[0,\infty)$ and, under ρ, $C[0,\infty)$ is a complete, separable metric space.

4.2 Problem. Let $\mathscr{C}(\mathscr{C}_t)$ be the collection of finite-dimensional cylinder sets of the form (2.1); i.e.,

$$C = \{\omega\in C[0,\infty); (\omega(t_1),\dots,\omega(t_n))\in A\}; \quad n\ge 1,\ A\in\mathscr{B}(\mathbb{R}^n), \tag{2.1$'$}$$

where, for all $i=1,\dots,n$, $t_i\in[0,\infty)$ (respectively, $t_i\in[0,t]$). Denote by $\mathscr{G}(\mathscr{G}_t)$ the smallest σ-field containing $\mathscr{C}(\mathscr{C}_t)$.

Show that $\mathscr{G} = \mathscr{B}(C[0,\infty))$, the Borel σ-field generated by the open sets in $C[0,\infty)$, and that $\mathscr{G}_t = \varphi_t^{-1}(\mathscr{B}(C[0,\infty))) \triangleq \mathscr{B}_t(C[0,\infty))$, where $\varphi_t\colon C[0,\infty)\to C[0,\infty)$ is the mapping $(\varphi_t\omega)(s) = \omega(t\wedge s)$; $0\le s<\infty$.

Whenever X is a random variable on a probability space $(\Omega,\mathscr{F},P)$ with values in a measurable space $(S,\mathscr{B}(S))$, i.e., the function $X\colon\Omega\to S$ is $\mathscr{F}/\mathscr{B}(S)$-measurable, then X induces a probability measure PX^{-1} on $(S,\mathscr{B}(S))$ by

$$PX^{-1}(B) = P\{\omega\in\Omega; X(\omega)\in B\}, \quad B\in\mathscr{B}(S). \tag{4.2}$$

An important special case of (4.2) occurs when $X=\{X_t; 0\le t<\infty\}$ is a continuous stochastic process on $(\Omega,\mathscr{F},P)$. Such an X can be regarded as a random variable on $(\Omega,\mathscr{F},P)$ with values in $(C[0,\infty),\mathscr{B}(C[0,\infty)))$, and PX^{-1} is called the *law of* X. The reader should verify that the law of a continuous process is determined by its finite-dimensional distributions.

A. Weak Convergence

The following concept is of fundamental importance in probability theory.

4.3 Definition. Let (S,ρ) be a metric space with Borel σ-field $\mathscr{B}(S)$. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of probability measures on $(S,\mathscr{B}(S))$, and let P be another measure on this space. We say that $\{P_n\}_{n=1}^{\infty}$ *converges weakly to* P and write $P_n \xrightarrow{w} P$, if and only if