$$d\tilde{\mathbb{P}} = \delta(\omega)\, d\mathbb{P}$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}(X \mid \mathcal{F}) = \mathbb{E}(X\,\delta \mid \mathcal{F})$$

$$\int_F \mathbb{E}(X\,\delta \mid \mathcal{F})\, d\mathbb{P} = \int_F X\,\delta\, d\mathbb{P} = \int_F X\, d\tilde{\mathbb{P}} = \int_F \mathbb{E}^{\tilde{\mathbb{P}}}(X \mid \mathcal{F})\, d\tilde{\mathbb{P}} = \int_F \mathbb{E}^{\tilde{\mathbb{P}}}(X \mid \mathcal{F})\,\delta\, d\mathbb{P}$$

$$= \int_F \mathbb{E}\big[\mathbb{E}^{\tilde{\mathbb{P}}}(X \mid \mathcal{F})\,\delta \mid \mathcal{F}\big]\, d\mathbb{P} = \int_F \mathbb{E}^{\tilde{\mathbb{P}}}(X \mid \mathcal{F})\,\mathbb{E}(\delta \mid \mathcal{F})\, d\mathbb{P}$$

$$\mathbb{E}(X\,\delta \mid \mathcal{F}) = \mathbb{E}^{\tilde{\mathbb{P}}}(X \mid \mathcal{F})\,\mathbb{E}(\delta \mid \mathcal{F})$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}(X \mid \mathcal{F}) = \frac{\mathbb{E}(X\,\delta \mid \mathcal{F})}{\mathbb{E}(\delta \mid \mathcal{F})}$$

$$\delta_T(\omega) = \exp\Big(\int_0^T f(r)dW_r - \frac{1}{2}\int_0^T f(r)^2\, dr\Big) \qquad X_t = g(t) + W_t$$

$$\delta_{s,t}(\omega) = \exp\Big(\int_s^t f(r)dW_r - \frac{1}{2}\int_s^t f(r)^2\, dr\Big)$$

$$\gamma_T(\omega) = \exp\Big(\int_0^T f(r)dW_r\Big) \qquad \gamma_{s,t}(\omega) = \exp\Big(\int_s^t f(r)dW_r\Big)$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}(X_t - X_s \mid \mathcal{F}_s) = g(t) - g(s) + \mathbb{E}^{\tilde{\mathbb{P}}}(W_t - W_s \mid \mathcal{F}_s)$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}(W_t - W_s \mid \mathcal{F}_s) = \frac{\mathbb{E}((W_t - W_s)\,\delta_T \mid \mathcal{F}_s)}{\mathbb{E}(\delta_T \mid \mathcal{F}_s)} = \frac{\mathbb{E}((W_t - W_s)\,\delta_T \mid \mathcal{F}_s)}{\delta_s}$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}(W_t - W_s \mid \mathcal{F}_s) = \mathbb{E}((W_t - W_s)\,\delta_{s,T} \mid \mathcal{F}_s) = \mathbb{E}((W_t - W_s)\,\delta_{s,T}) = \mathbb{E}((W_t - W_s)\,\delta_{s,t})\mathbb{E}(\delta_{t,T})$$

$$\mathbb{E}((W_t - W_s)\,\delta_{s,t}) = \mathbb{E}((W_t - W_s)\,\gamma_{s,t})\ \exp\Big(-\frac{1}{2}\int_s^t f(r)^2\, dr\Big)$$

$$\mathbb{E}(\delta_{t,T}) = \mathbb{E}(\gamma_{t,T})\ \exp\Big(-\frac{1}{2}\int_t^T f(r)^2\, dr\Big) = \exp\Big(\frac{1}{2}\int_t^T f(r)^2\, dr\Big)\ \exp\Big(-\frac{1}{2}\int_t^T f(r)^2\, dr\Big) = 1$$

$$\mathbb{E}((W_t - W_s)\,\gamma_{s,t}) = \frac{d}{d\alpha}\mathbb{E}\Big[\exp\Big(\alpha(W_t - W_s) + \int_s^t f(r)dW_r\Big)\Big]_{\alpha=0}$$

$$\mathbb{E}((W_t - W_s)\,\gamma_{s,t}) = \frac{d}{d\alpha}\mathbb{E}\Big[\exp\Big(\int_s^t (\alpha + f(r))dW_r\Big)\Big]_{\alpha=0} = \frac{d}{d\alpha}\Big[\exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big)\Big]_{\alpha=0}$$

$$\mathbb{E}((W_t - W_s)\,\gamma_{s,t}) = \frac{d}{d\alpha}\Big[\exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big)\Big]_{\alpha=0} = \int_s^t f(r)dr\ \exp\Big(\frac{1}{2}\int_s^t f(r)^2 dr\Big)$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}(X_t - X_s \mid \mathcal{F}_s) = g(t) - g(s) + \int_s^t f(r)dr$$

Scegliendo

$$g(t) = -\int_0^t f(r)\,dr$$

abbiamo che $\mathbb{E}^{\tilde{\mathbb{P}}}(X_t - X_s \mid \mathcal{F}_s) = 0$ cioè X_t è una martingala.

$$\mathbb{E}^{\tilde{\mathbb{P}}}((X_t - X_s)^2 \mid \mathcal{F}_s) = \mathbb{E}^{\tilde{\mathbb{P}}}((W_t - W_s)^2 \mid \mathcal{F}_s) - (g(t) - g(s))^2$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}((W_t - W_s)^2 \mid \mathcal{F}_s) = \mathbb{E}((W_t - W_s)^2\,\delta_{s,t})$$

$$\mathbb{E}((W_t - W_s)^2\,\delta_{s,t}) = \mathbb{E}((W_t - W_s)^2\,\gamma_{s,t})\ \exp\Big(-\frac{1}{2}\int_s^t f(r)^2\,dr\Big)$$

$$\mathbb{E}((W_t - W_s)^2\,\gamma_{s,t}) = \frac{d^2}{d\alpha^2}\mathbb{E}\Big[\exp\Big(\alpha(W_t - W_s) + \int_s^t f(r)dW_r\Big)\Big]_{\alpha=0}$$

$$\mathbb{E}((W_t - W_s)^2\,\gamma_{s,t}) = \frac{d^2}{d\alpha^2}\mathbb{E}\Big[\exp\Big(\int_s^t (\alpha + f(r))dW_r\Big)\Big]_{\alpha=0} = \frac{d^2}{d\alpha^2}\Big[\exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big)\Big]_{\alpha=0}$$

$$\frac{d}{d\alpha}\exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big) = \int_s^t (\alpha + f(r))dr\ \exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big)$$

$$\frac{d^2}{d\alpha^2}\exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big) =$$

$$(t-s)\ \exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big) + \Big(\int_s^t (\alpha + f(r))dr\Big)^2\ \exp\Big(\frac{1}{2}\int_s^t (\alpha + f(r))^2 dr\Big)$$

$$\mathbb{E}((W_t - W_s)^2\,\gamma_{s,t}) = \Bigg[(t-s) + \Big(\int_s^t f(r)\,dr\Big)^2\Bigg]\exp\Big(\frac{1}{2}\int_s^t f(r)^2 dr\Big)$$

$$\mathbb{E}^{\tilde{\mathbb{P}}}((X_t - X_s)^2 \mid \mathcal{F}_s) = t - s$$

Quindi $X_t^2 - t$ è una martingala da cui per un teorema di Lévy abbiamo che

$$X_t = W_t - \int_0^t f(r)\,dr$$

è un processo di Wiener rispetto a $\tilde{\mathbb{P}}$.