$$p(s,x;t,I) := \mathbb{P}(X_t \in I \mid X_s = x)$$

Processo di Markov
Processo di diffusione

$$\frac{1}{t-s}\mathbb{P}\{X_t \in B(x,\varepsilon)^c \mid X_s = x\} = \frac{1}{t-s}\int_{|y-x|>\varepsilon} p(s,x;t,dy) = \frac{p(s,x;t,B(x,\varepsilon)^c)}{t-s} \to 0$$

- $\mathbb{P}\{X_t \in B(x,\varepsilon)^c \mid X_s = x\} = o(t-s)$
- $\displaystyle\frac{1}{t-s}\int_{B(x,\varepsilon)} (y-x)\, p(s,x;t,dy) \to b(s,x)$
- $\displaystyle\frac{1}{t-s}\int_{B(x,\varepsilon)} (y-x)^2\, p(s,x;t,dy) \to a(s,x)$

---

densità $p(s,x;t,y)$

- $\displaystyle\frac{1}{t-s}\int_{|y-x|>\varepsilon} p(s,x;t,y)\,dy \to 0 \quad \text{as } t \searrow s$
- $\displaystyle\frac{1}{t-s}\int_{B(x,\varepsilon)} (y-x)\, p(s,x;t,y)\,dy \to b(s,x)$
- $\displaystyle\frac{1}{t-s}\int_{B(x,\varepsilon)} (y-x)^2\, p(s,x;t,y)\,dy \to a(s,x)$

---

omogeneità temporale $p(s,x;t,I) = p(t-s,x,I) \qquad \tau = t-s$

- $\displaystyle\frac{1}{\tau}\int_{|y-x|>\varepsilon} p(\tau,x,dy) \to 0 \quad \text{as } \tau \searrow 0$
- $\displaystyle\frac{1}{\tau}\int_{B(x,\varepsilon)} (y-x)\, p(\tau,x,dy) \to b(x)$
- $\displaystyle\frac{1}{\tau}\int_{B(x,\varepsilon)} (y-x)^2\, p(\tau,x,dy) \to a(x)$

---

omogeneità temporale e densità $p(t-s,x,y) \qquad \tau = t-s$

- $\displaystyle\frac{1}{\tau}\int_{|y-x|>\varepsilon} p(\tau,x,y)\,dy \to 0 \quad \text{as } \tau \searrow 0$

$$\bullet\ \frac{1}{\tau}\int_{B(x,\varepsilon)}(y-x)\,p(\tau,x,y)\,dy\to b(x)$$

$$\bullet\ \frac{1}{\tau}\int_{B(x,\varepsilon)}(y-x)^2\,p(\tau,x,y)\,dy\to a(x)$$

---

omogeneità temporale e densità $p(t-s,x,y)$ $\qquad \tau=t-s$

$$\bullet\ \frac{1}{\tau}\int_{|y-x|>\varepsilon}p(\tau,x,y)\,dy\to 0 \quad \text{as } \tau\searrow 0$$

$$\bullet\ \frac{1}{\tau}\int_{B(x,\varepsilon)}(y-x)\,p(\tau,x,y)\,dy\to b(x)$$

$$\bullet\ \frac{1}{\tau}\int_{B(x,\varepsilon)}(y-x)^2\,p(\tau,x,y)\,dy\to a(x)$$

---

omogeneità temporale, omegeneità spaziale e densità $p_{t-s}(y-x)$
$\tau=t-s$, $\xi=y-x$

$$\bullet\ \frac{1}{\tau}\int_{|\xi|>\varepsilon}p_\tau(\xi)\,d\xi\to 0 \quad \text{as } \tau\searrow 0$$

$$\bullet\ \frac{1}{\tau}\int_{|\xi|\le\varepsilon}\xi\,p_\tau(\xi)\,d\xi\to b \qquad \text{una costante}$$

$$\bullet\ \frac{1}{\tau}\int_{|\xi|\le\varepsilon}\xi^2\,p_\tau(\xi)\,d\xi\to a \qquad \text{una costante}$$

$$u(s,x) = \int_{\mathbb{R}} f(y)\, p(s,x;t,dy) = \mathbb{E}(f(X_t) \mid X_s = x), \qquad 0 < s < t$$

$$u(t,x) = f(x)$$

$$-\frac{\partial u}{\partial s} = \tfrac{1}{2}\, a(s,x)\, \frac{\partial^2 u}{\partial x^2} + b(s,x)\, \frac{\partial u}{\partial x}$$

$u(s,x) = u(s,x;t)$

$$u(s,x) = \int_{\mathbb{R}} f(y)\, p(s,x;t,y)\, dy = \mathbb{E}(f(X_t) \mid X_s = x), \qquad 0 < s < t$$

$$u(t,x) = f(x)$$

$$-\frac{\partial u}{\partial s} = \tfrac{1}{2}\, a(s,x)\, \frac{\partial^2 u}{\partial x^2} + b(s,x)\, \frac{\partial u}{\partial x}$$

$$v(t,y) = \int_{\mathbb{R}} g(x)\, p(s,x;t,y)\, dx, \qquad t > s$$

$u(t,y) = v(s;t,y)$

$$v(s,y) = g(y)$$

$$\frac{\partial v}{\partial t} = \tfrac{1}{2}\, \frac{\partial^2}{\partial y^2}(a(t,y)\, v) - \frac{\partial}{\partial y}(b(t,y)\, v)$$

$$u(\tau,x) = \int_{\mathbb{R}} f(y)\, p(\tau,x,y)\, dy = \mathbb{E}(f(X_\tau) \mid X_0 = x), \qquad 0 < \tau$$

$$u(0,x) = f(x)$$

$$\frac{\partial u}{\partial \tau} = \tfrac{1}{2}\, a(x)\, \frac{\partial^2 u}{\partial x^2} + b(x)\, \frac{\partial u}{\partial x}$$

$$v(\tau,y) = \int_{\mathbb{R}} g(x)\, p(\tau,x,y)\, dx$$

$$v(0,y) = g(y)$$

$$\frac{\partial v}{\partial \tau} = \tfrac{1}{2}\, \frac{\partial^2}{\partial y^2}(a(y)\, v) - \frac{\partial}{\partial y}(b(y)\, v)$$

$$u(\tau, x) = \int_{\mathbb{R}} f(y)\, p_\tau(y - x)\, dy = (f * p_\tau)(x) = \mathbb{E}(f(X_\tau) \mid X_0 = x)$$

$$u(0, x) = f(x)$$

$$\frac{\partial u}{\partial \tau} = \tfrac{1}{2}\, a\, \frac{\partial^2 u}{\partial x^2} + b\, \frac{\partial u}{\partial x}$$