$$X\colon \Omega \to \mathbb{R}^n$$
$$Y\colon \Omega \to \mathbb{R}^m$$
$$(X,Y)\colon \Omega \to \mathbb{R}^{n+m}$$
$$X = (X_1, X_2, \dots, X_n)$$
$$X_k\colon \Omega \to \mathbb{R}$$

Caso di misure assolutamente continue rispetto alla misura di Lebesgue

$$\mu_{(X,Y)}(B) = \mathbb{P}((X,Y) \in B) = \int_B f(x,y)\,dxdy$$
$$B = J \times I$$
$$\mathbb{P}(X \in J, Y \in I) = \int_{J\times I} f(x,y)\,dxdy = \int_J \Big(\int_I f(x,y)\,dy\Big)\,dx$$
$$\varphi(x,y) = \frac{f(x,y)}{f_1(x)}$$
$$f_1(x) := \int_{\mathbb{R}^m} f(x,y)\,dy$$
$$\mathbb{P}(X \in J, Y \in I) = \int_J \Big(\int_I \varphi(x,y)\,dy\Big)\,f_1(x)dx$$
$$p(x,I) := \int_I \varphi(x,y)\,dy$$
$$\mathbb{P}(X \in J, Y \in I) = \int_J p(x,I)\,\mu_X(dx)$$
$$\boxed{\mathbb{P}(Y \in I \mid X = x) := p(x,I)}$$
$$\mathbb{P}(X \in J, Y \in I) = \int_J \mathbb{P}(Y \in I \mid X = x)\,\mathbb{P}(X \in dx)$$
$$\mathbb{P}(Y \in I) = \mathbb{P}(X \in \mathbb{R}^n, Y \in I) = \int_{\mathbb{R}^n} \mathbb{P}(Y \in I \mid X = x)\,\mathbb{P}(X \in dx)$$
$$\mathbb{P}(Y \in I \mid X \in J) = \frac{\mathbb{P}(X \in J, Y \in I)}{\mathbb{P}(X \in J)} = \frac{\displaystyle\int_J \mathbb{P}(Y \in I \mid X = x)\,\mathbb{P}(X \in dx)}{\displaystyle\int_J \mathbb{P}(X \in dx)}$$

$$\mathbb{P}(X_t \in I) = \mathbb{P}(X_s \in \mathbb{R}, X_t \in I) = \int_{\mathbb{R}} \mathbb{P}(X_t \in I \mid X_s = x)\, \mathbb{P}(X_s \in dx)$$

$$\boxed{p(s,x;t,I) := \mathbb{P}(X_t \in I \mid X_s = x)}$$

$$\boxed{\mu_t(I) = \int_R \mu_s(dx)\, p(s,x;t,I)}$$

$s < r < t$

$$\mu_t(I) = \int_R \mu_r(dz)\, p(r,z;t,I) = \int_R \Big(\int_{\mathbb{R}} \mu_s(dx)\, p(s,x;r,dz) \Big)\, p(r,z;t,I)$$

$$\mu_t(I) = \int_R \mu_s(dx) \underbrace{\Big(\int_{\mathbb{R}} p(s,x;r,dz)\, p(r,z;t,I) \Big)}_{\pi(s,x;r;t,I)}$$

$$\mathbb{P}(X \in J, Y \in I) = \int_J \mathbb{P}(Y \in I \mid X = x)\, \mathbb{P}(X \in dx)$$

$$\int_{\mathbb{R}^m} \psi(y)\, \mathbb{P}(X \in J, Y \in dy) = \int_{\mathbb{R}^m} \int_J \psi(y)\, \mathbb{P}(Y \in dy \mid X = x)\, \mathbb{P}(X \in dx)$$

$$\mathbb{P}(X \in dx, Y \in dy) = P(Y \in dy \mid X = x)\, \mathbb{P}(X \in dx)$$

$$\mathbb{P}(X \in dx) = \mathbb{P}(X_1 \in dx_1, X_2 \in dx_2, \dots, X_n \in dx_n)$$

$$X^{(n)} = (X_1, X_2, \dots, X_n) = X, \qquad x^{(n)} = (x_1, x_2, \dots, x_n) \qquad dx^{(n)} = dx_1 dx_2 \cdots dx_n$$

$$\mathbb{P}(X^{(n)} \in dx) = \mathbb{P}(X_n \in dx_n \mid X^{(n-1)} = x^{(n-1)})\, \mathbb{P}(X^{(n-1)} \in dx^{(n-1)})$$

$$\mathbb{P}(X_n \in dx_n \mid X^{(n-1)} = x^{(n-1)})\, \mathbb{P}(X_{n-1} \in dx_{n-1} \mid X^{(n-2)} = x^{(n-2)})\, \mathbb{P}(X^{(n-2)} \in dx^{(n-2)})$$

$$\begin{aligned}\mathbb{P}(X_1 \in dx_1, X_2 \in dx_2, X_3 \in dx_3) = \mathbb{P}(X_3 \in dx_3 \mid X_2 = x_2, X_1 = x_1)\\ \cdot\mathbb{P}(X_2 \in dx_2 \mid X_1 = x_1)\\ \cdot\mathbb{P}(X_1 \in dx_1)\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^3} f(x_1,x_2,x_3)\mathbb{P}(X_1 \in dx_1, X_2 \in dx_2, X_3 \in dx_3) = \int_{\mathbb{R}} \mathbb{P}(X_1 \in dx_1)\\ \int_{\mathbb{R}} \mathbb{P}(X_2 \in dx_2 \mid X_1 = x_1)\\ \int_{\mathbb{R}} f(x_1,x_2,x_3)\mathbb{P}(X_3 \in dx_3 \mid X_2 = x_2, X_1 = x_1)\end{aligned}$$

$$\mathbb{P}(X_1 \in dx_1, X_2 \in dx_2) = \int_{\mathbb{R}} \mathbb{P}(X_2 \in dx_2 \mid X_1 = x_1)\, \mathbb{P}(X_1 \in dx_1)$$

$$\mathbb{P}(X_2 \in dx_2, X_3 \in dx_3) = \int_{\mathbb{R}} \mathbb{P}(X_3 \in dx_3 \mid X_2 = x_2)\, \mathbb{P}(X_2 \in dx_2)$$

$$\mathbb{P}(X_1 \in dx_1, X_3 \in dx_3) = \int_{\mathbb{R}} \overbrace{\mathbb{P}(X_3 \in dx_3 \mid X_1 = x_1)}^{p(x_1, dx_3)}\, \mathbb{P}(X_1 \in dx_1)$$

$$\int_{\mathbb{R}} \mathbb{P}(X_3 \in I \mid X_2 = x_2)\, \mathbb{P}(X_2 \in dx_2 \mid X_1 = x_1) = \pi(x_1, I)$$

$$\int_{\mathbb{R}} \mathbb{P}(X_3 \in dx_3 \mid X_2 = x_2)\, \mathbb{P}(X_2 \in dx_2 \mid X_1 = x_1) = \pi(x_1, dx_3)$$

In generale $p(x_1, dx_3) \neq \pi(x_1, dx_3)$!

Caso Gaussiano $m(t)$, $a(s,t)$, $s < t$.

$$p(s, x; t, dy) := \mathbb{P}(X_t \in dy \mid X_s = x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\, dy =: p(x, dy)$$

dove

$$\mu = m(t) + \frac{a(s,t)}{a(s,s)}\big(x - m(s)\big), \qquad \sigma^2 = \frac{a(s,s)a(t,t) - a(s,t)^2}{a(s,s)}$$

$s < r < t$.

$$\mathbb{P}(X_r \in dz \mid X_s = x) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(z-\mu_1)^2}{2\sigma_1^2}}\, dz$$

$$\mu_1 = m(r) + \frac{a(s,r)}{a(s,s)}\big(x - m(s)\big), \qquad \sigma_1^2 = \frac{a(s,s)a(r,r) - a(s,r)^2}{a(s,s)}$$

$$\mathbb{P}(X_t \in dy \mid X_r = z) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}\, dy$$

$$\mu_2 = m(t) + \frac{a(r,t)}{a(r,r)}\big(z - m(r)\big), \qquad \sigma_2^2 = \frac{a(r,r)a(t,t) - a(r,t)^2}{a(r,r)}$$

$$\pi(x, dy) = \int_{\mathbb{R}} \mathbb{P}(X_t \in dy \mid X_r = z)\, \mathbb{P}(X_r \in dz \mid X_s = x) =$$

$$= \Big[\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}} \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{(z-\mu_1)^2}{2\sigma_1^2}}\, dz \Big]\, dy$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\, dy$$

$$\mu = \frac{a(s,r)a(r,t)}{a(s,s)a(r,r)}\big(x - m(s)\big) + m(t)$$

$$\sigma^2 = a(t,t) - \frac{a(r,t)^2}{a(r,r)^2}\frac{a(s,r)^2}{a(s,s)} = \left(\frac{a(r,t)}{a(r,r)}\right)^2 \underbrace{\left[a(r,r) - \frac{a(s,r)^2}{a(s,s)}\right]}_{>0} + \underbrace{a(t,t) - \frac{a(r,t)^2}{a(r,r)}}_{>0}$$

Affinché $\pi(x,dy) = p(x,dy)$ deve risultare

$$\frac{a(s,r)a(r,t)}{a(s,s)a(r,r)}\big(x - m(s)\big) + m(t) = m(t) + \frac{a(s,t)}{a(s,s)}\big(x - m(s)\big)$$

$$\boxed{a(s,t) = \frac{a(s,r)a(r,t)}{a(r,r)}}$$

$$a(t,t) - \frac{a(s,t)^2}{a(s,s)} = a(t,t) - \frac{a(s,r)^2 a(r,t)^2}{a(s,s)a(r,r)^2}$$

Equazione (relazione) di Chapman-Kolmogorov

$s < r < t$

$$p(s,x;t,I) = \int_{\mathbb{R}} \underbrace{p(s,x;r,dz)}_{\nu(dz)}\ \underbrace{p(r,z;t,I)}_{\varphi(z)}$$

Nel caso Gaussiano la relazione di Chapman-Kolmogorov è verificata se e solo se

$$\boxed{a(s,t) = \frac{a(s,r)a(r,t)}{a(r,r)}}$$

Esempi: Processo di Wiener ($a(s,t) = s \wedge t$)
Brownian bridge ($a(s,t) = s \wedge t - s\,t$)

Fractional Brownian motion with Hurst index α ($0 < \alpha \leq 1$)

$$a^{(\alpha)}(s,t) = \tfrac{1}{2}\left(|s|^{2\alpha} + |t|^{2\alpha} - |t-s|^{2\alpha}\right)$$

Per $\alpha = \frac{1}{2}$ risulta $a^{(1/2)}(s,t) = s \wedge t$!

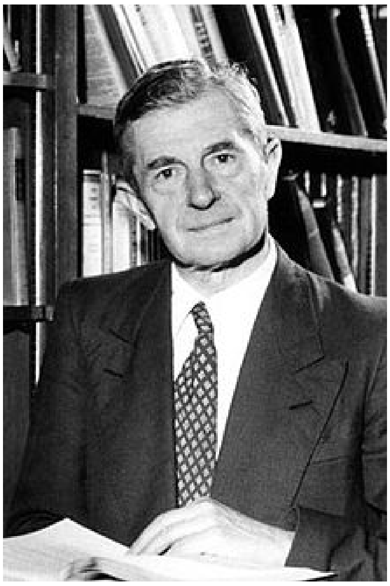 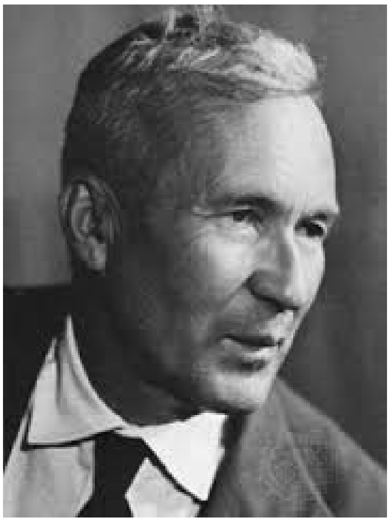

Figure 1: Sidney Chapman (1888-1970) and Andrej Kolmogorov (1903-1987)

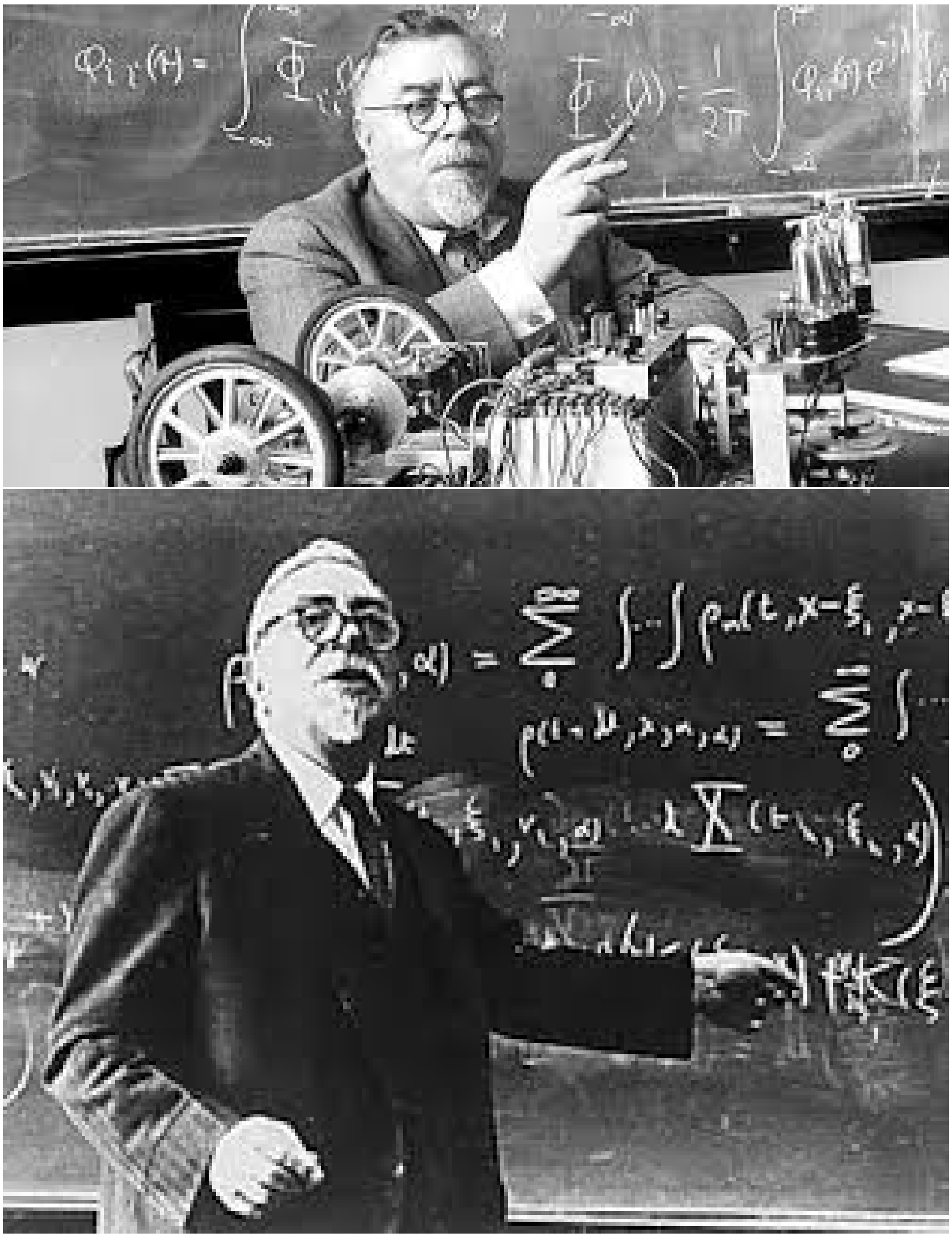

Figure 2: Norbert Wiener (1894-1964)

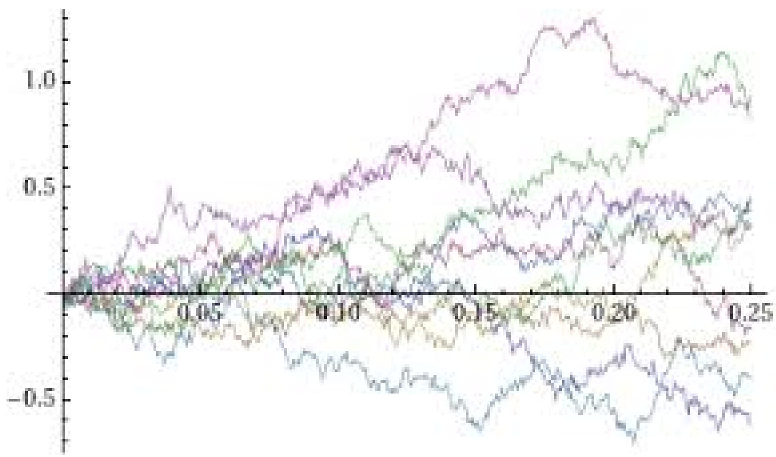

Figure 3: typical Wiener trajectories