Valore di aspettazione condizionato

$$\mathbb{E}(XY|\mathcal{F}) = Y\,\mathbb{E}(X|\mathcal{F})$$

se Y è $\mathcal{F}$-misurabile.

$X \in L^1$

$XY \in L^1$

Dim. $Y = \mathbb{1}_{F_0}$ $\qquad F_0 \in \mathcal{F}$

$\mathbb{E}(XY|\mathcal{F})$ $\quad \mathcal{F}$-misurabile

$$\int_F XY\,dP = \int_F \mathbb{E}(XY|\mathcal{F})\,dP \qquad \forall F \in \mathcal{F}$$

$$\| $$

$$\int_F X\mathbb{1}_{F_0}\,dP = \int_{F\cap F_0} X\,dP = \int_{F\cap F_0} \mathbb{E}(X|\mathcal{F})\,dP =$$

$$= \int_F \mathbb{1}_{F_0} \cdot \mathbb{E}(X|\mathcal{F})\,dP$$

Per linearità è vero anche per funzioni semplici

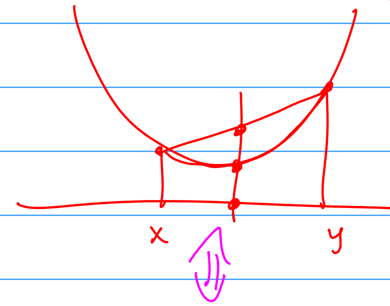
$$\sum \alpha_i \mathbb{1}_{F_i}$$

$Y \geq 0 \quad Y_n \uparrow Y$ $\qquad$ ——— $\qquad Y = Y^+ - Y^-$

Disuguaglianza di Jensen

$\phi : \mathbb{R} \to \mathbb{R}$ convessa

$$\phi\big((1-\lambda)x + \lambda y\big) \le (1-\lambda)\phi(x) + \lambda\phi(y)$$

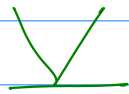


$$\phi\big(\mathbb{E}(X)\big) \le \mathbb{E}\big(\phi(X)\big)$$

$$\phi\Big(\sum_i \lambda_i x_i\Big) \le \sum_i \lambda_i \phi(x_i)$$

$\lambda_i \ge 0 \quad \sum \lambda_i = 1$

$\phi(x) = |x|$



$$|\mathbb{E}(X)| \le \mathbb{E}(|X|)$$

$$\phi\Big(\int_\Lambda x(\lambda)\,\mu(d\lambda)\Big) \le \int_\Lambda \phi(x(\lambda))\,\mu(d\lambda)$$

$\mu \ge 0 \quad \int_\Lambda \mu(d\lambda) = 1$

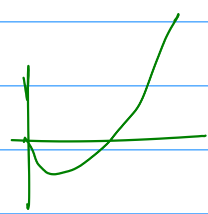
$|x|^p \qquad p \ge 1$

$\| \qquad$

$e^{\frac{p}{2}\log|x|^2} \qquad \phi'' \ge 0$

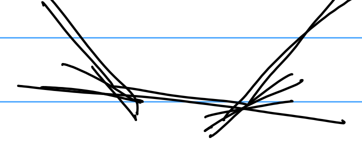
$$|\mathbb{E}(X)| \le \mathbb{E}\big(|X|^p\big)^{1/p}$$

$e^{\alpha x}, \quad x\log x$



$$\phi\big(\mathbb{E}(X|\mathcal{F})\big) \le \mathbb{E}\big(\phi(X)|\mathcal{F}\big)$$

Dimostr. ϕ è convessa $\quad \sup(a+bx) = \phi(x)$

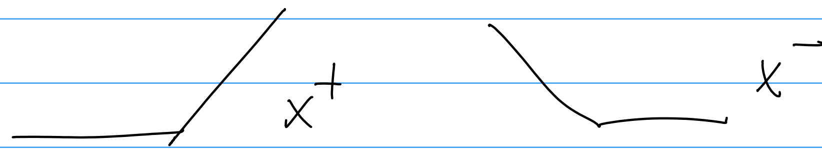


$a + bX \le \phi(X)$

$$a + b\,\mathbb{E}(X|\mathcal{F}) = \mathbb{E}(a + bX \,|\, \mathcal{F}) \leq \mathbb{E}(\phi(X)|\mathcal{F})$$

$$\phi(\mathbb{E}(X|\mathcal{F})) \leq \mathbb{E}(\phi(X)|\mathcal{F})$$

$$\underbrace{\phi(x) = |x|^p}$$



$$|\mathbb{E}(X|\mathcal{F})|^p \leq \mathbb{E}(|X|^p|\mathcal{F})$$

$F \in \mathcal{F}$

$$\int_F |\mathbb{E}(X|\mathcal{F})|^p \, dP \leq \int_F \mathbb{E}(|X|^p|\mathcal{F}) \, dP$$

$$= \int_F |X|^p \, dP$$

$\{X_t\}$ processo che sia una martingala rispetto ad una filtrazione $(\mathcal{F}_t)$

$$\mathbb{E}(X_t \,|\, \mathcal{F}_s) = X_s \qquad \text{q.o.} \qquad \forall s \leq t$$

$|X_t|^p$

$$\mathbb{E}(|X_t|^p \,|\, \mathcal{F}_s) = |X_s|^p$$

$$\geq \left| \mathbb{E}(X_t \mid \mathcal{F}_s) \right|^p = |X_s|^p$$

$$\underbrace{\int_F |X_s|^p \, dP} \leq \int_F \mathbb{E}\left(|X_t|^p \mid \mathcal{F}_s \right) dP$$

$F \in \mathcal{F}_s$

$$\leq \quad = \underbrace{\int_F |X_t|^p \, dP}$$

Applicazione X_t sia una martingala tale sia continua a destra, allora

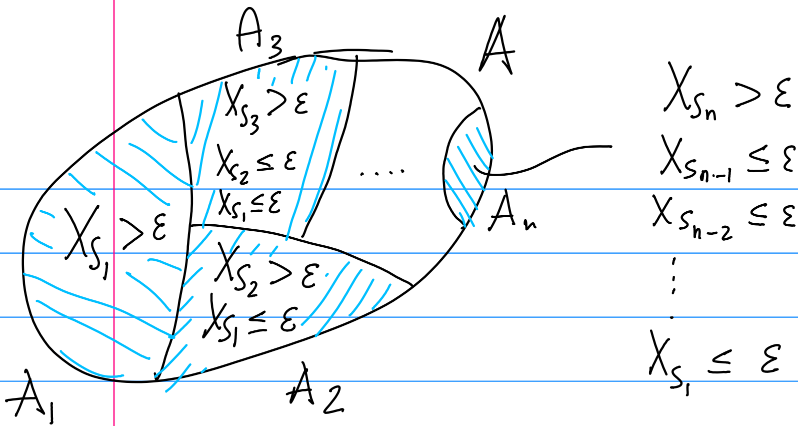
$$P\left(\max_{0 \leq s \leq t} |X_s| > \varepsilon \right) \leq \frac{1}{\varepsilon^p} \mathbb{E}\left(|X_t|^p\right)$$

$p \geq 1$

Dimostrazione. $0 \leq s_1 \leq s_2 \leq s_3 \leq \dots \leq s_n \leq t$

$$P\left(\max_i |X_{s_i}| \geq \varepsilon \right) \leq \frac{1}{\varepsilon^p} \mathbb{E}\left(|X_t|^p \right)$$

$$A = \left(\max_i |X_{s_i}| \geq \varepsilon \right)$$



$$A = A_1 \cup A_2 \cup A_3 \cup \cdots\cdots \cup A_n$$

$$P(A) = \sum_i P(A_i)$$

$$P(A_i) = P\left(|X_{s_i}| > \varepsilon, |X_{s_{i-1}}| \le \varepsilon, \ldots, |X_{s_1}| \le \varepsilon\right)$$

$$\le \int_{(|X_{s_i}| > \varepsilon, \ldots\ldots)} \frac{|X_{s_i}|^p}{\varepsilon^p} \, dP = \frac{1}{\varepsilon^p} \int_{A_i} |X_{s_i}|^p \, dP \le \frac{1}{\varepsilon^p} \int_{A_i} |X_t|^p \, dP$$

martingale

$A_i \in \mathcal{F}_{s_i}$

$$P(A) \le \frac{1}{\varepsilon^p} \int_A |X_t|^p \, dP$$

$(0 \le s \le t)$ $\qquad$ $(0 \le s \le t) \cap \mathbb{Q}$

$$\bigcup_n \{\underbrace{0 \le q \le t \quad q = \tfrac{\alpha}{\beta} \quad |\beta| \le n}_{C_n}\}$$

$$P\left(\max_q |X_q| \ge \varepsilon, \ q \in C_n\right)$$

$$\leq \frac{1}{\varepsilon^p} \mathbb{E}\left(|X_t|^p\right)$$

$$P\left(\max_{s} |X_s| \geq \varepsilon, \quad s \in [0,t] \cap \mathbb{Q}\right) \leq \frac{1}{\varepsilon^p} \mathbb{E}\left(|X_t|^p\right)$$

Applicazione: W_t è continuo ed è una martingala

$$P\left(\max_{0 \leq s \leq t} |W_s| \geq \varepsilon\right) \leq \frac{1}{\varepsilon} \int_\Omega |W_t| \, dP$$

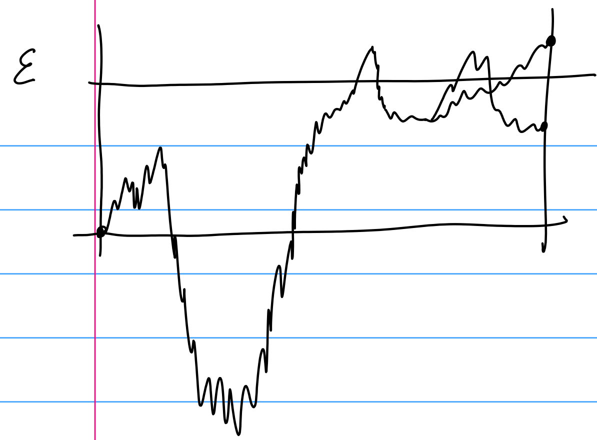
$$\frac{x}{\sqrt{t}} = y$$

$$\int_\Omega |W_t| \, dP = \int_{\mathbb{R}} |x| \frac{e^{-x^2/2t}}{\sqrt{2\pi t}} \, dx =$$

$$= \int \sqrt{t} \, |y| \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy = C \sqrt{t}$$

$$P\left(\max_{0 \leq s \leq t} |W_s| \geq \varepsilon\right) \leq \frac{C\sqrt{t}}{\varepsilon}$$

$$P\left(\max_{0 \leq s \leq t} |W_s| \geq \varepsilon\right)$$



$$P\left(\max_{0\le s\le t} W_s \ge \varepsilon\right)$$

$$\left(\max_{0\le s\le t} W_s \ge \varepsilon\right) = \left(W_t \ge \varepsilon\right)$$

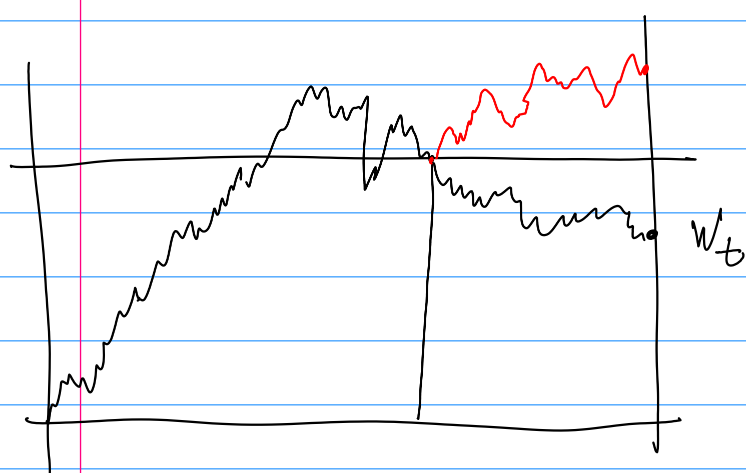
$$\cup \left(W_t < \varepsilon, \max_{0\le s\le t} W_s \ge \varepsilon\right)$$

$$P\left(\max_{0\le s\le t} W_s \ge \varepsilon\right) =$$

$$= P\left(W_t \ge \varepsilon\right) + P\left(\max_{0\le s\le t} W_s \ge \varepsilon, W_t < \varepsilon\right)$$

$$\|$$

$$P\left(W_t > \varepsilon\right)$$



$$= 2\int_{\varepsilon}^{+\infty} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2t}}\,dx$$

$$\frac{x}{\sqrt{t}} =: y$$

$$= 2\int_{\varepsilon/\sqrt{t}}^{+\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy \qquad \frac{\sqrt{t}}{\varepsilon}$$

$$2\int_{\varepsilon/\sqrt{t}}^{+\infty} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy = \frac{2}{\sqrt{2\pi}} \int_{\varepsilon/\sqrt{t}}^{+\infty} 1\, e^{-y^2/2}\,dy$$

$$\leq \frac{2}{\sqrt{2\pi}} \cdot \frac{\sqrt{t}}{\varepsilon} \int_{\varepsilon/\sqrt{t}}^{+\infty} y\, e^{-y^2/2}\,dy \qquad -e^{-y^2/2}$$

$$= \frac{2}{\sqrt{2\pi}} \frac{\sqrt{t}}{\varepsilon} e^{-\frac{\varepsilon^2}{2t}}$$

$$\mathbb{E}\left(e^{\alpha W_t - \frac{\alpha^2}{2}t} \,\middle|\, \mathcal{F}_s\right) = e^{\alpha W_s - \frac{\alpha^2 s}{2}}$$

$$\parallel$$

$$e^{-\frac{\alpha^2}{2}t}\, \mathbb{E}\left(e^{\alpha W_t} \,\middle|\, \mathcal{F}_s\right)$$

$$= e^{-\frac{\alpha^2}{2}t}\, \mathbb{E}\left(\sum_{m=0}^{\infty} \frac{\alpha^m W_t^m}{m!} \,\middle|\, \mathcal{F}_s\right) =$$

$$= e^{-\frac{\alpha^2}{2}t} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!}\, \mathbb{E}\left(W_t^m \,\middle|\, \mathcal{F}_s\right)$$

$s<t$

$$\mathbb{E}\left(W_t^m \mid \mathcal{F}_s\right) =$$

$$= \mathbb{E}\left(\left(W_t - W_s + W_s\right)^m \mid \mathcal{F}_s\right) =$$

$$= \sum_{k=0}^{m} \binom{m}{k} \mathbb{E}\left(\left(W_t - W_s\right)^k W_s^{m-k} \mid \mathcal{F}_s\right)$$

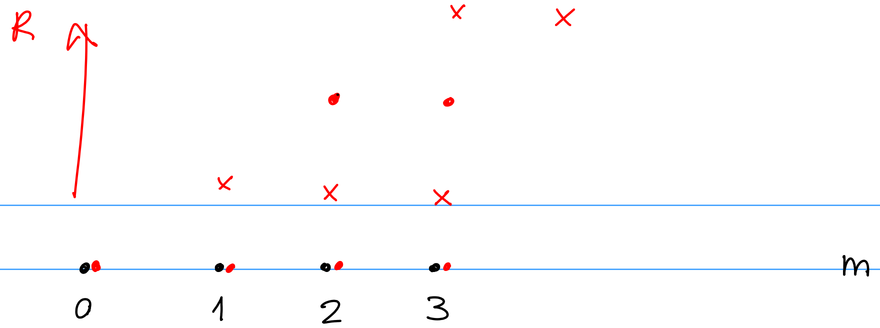
$$= \sum_{k=0}^{m} \binom{m}{k} W_s^{m-k} \mathbb{E}\left(\left(W_t - W_s\right)^k\right)$$

$$\mathbb{E}\left(\left(W_t - W_s\right)^k\right) = \int_{\mathbb{R}} y^k \frac{e^{-\frac{y^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} dy$$

$$x = \frac{y}{\sqrt{t-s}}$$

$$= \begin{cases} = 0 & k \text{ dispari} \\ & = (t-s)^{k/2} \underbrace{\int_{\mathbb{R}} x^k \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx}_{(k-1)!!} \end{cases}$$

$$= e^{-\frac{\alpha^2}{2}t} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} \sum_{\substack{k=0 \\ k \text{ pari}}}^{m} \binom{m}{k} W_s^{m-k} (k-1)!! (t-s)^{k/2}$$



$$= e^{-\frac{\alpha^2}{2}t} \sum_{\substack{k=0 \\ k \text{ pari}}}^{\infty} \sum_{m=k}^{\infty} \frac{\alpha^m}{m!} \binom{m}{k} W_s^{m-k} (k-1)!! \, (t-s)^{k/2} =$$

$$= e^{-\frac{\alpha^2}{2}t} \sum_{\substack{k=0 \\ k \text{ pari}}}^{\infty} \frac{\alpha^k}{k!} (t-s)^{k/2} (k-1)!! \underbrace{\sum_{m=k}^{\infty} \frac{\alpha^{m-k}}{(m-k)!} W_s^{m-k}}_{e^{\alpha W_s}}$$

$$= e^{-\frac{\alpha^2}{2}t} e^{\alpha W_s} \sum_{\substack{k=0 \\ k \text{ pari}}}^{\infty} \frac{\alpha^k}{k!} (t-s)^{k/2} (k-1)!! =$$

$$= e^{-\frac{\alpha^2}{2}t + \alpha W_s} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(2k)!} (t-s)^k \frac{(2k)!}{2^k k!}$$

$$= e^{-\frac{\alpha^2}{2}t + \alpha W_s} \sum_{k=0}^{\infty} \left[\frac{\alpha^2 (t-s)}{2}\right]^k \Big/ k! = e^{\alpha W_s - \frac{\alpha^2 s}{2}}$$

$$\mathbb{E}\left(e^{\alpha W_t - \frac{\alpha^2 t}{2}}\right) = 1$$

$$\int_\Omega \underbrace{e^{\alpha W_t(\omega) - \frac{\alpha^2 t}{2}} \mathbb{P}(d\omega)}_{\mathbb{P}'(d\omega)} = 1$$

$$\alpha W_t - \alpha t \qquad \mathbb{P}(d\omega) \qquad \mathbb{P}'(d\omega)$$

"<u>Trasformata di Girsanov</u>".

(X_t) e' una $(\mathcal{F}_t)$-martingala

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \quad \text{q.o.} \qquad s \le t$$

1. Conseg. $\mathbb{E}(X_s) = \mathbb{E}\left(\mathbb{E}(X_t \mid \mathcal{F}_s)\right) = \mathbb{E}(X_t)$

$$\mathring{\mathcal{F}}_t \subset \mathcal{F}_t \qquad \mathring{\mathcal{F}}_t \subset \mathcal{G}_t \subset \mathcal{F}_t$$

$$\mathbb{E}(X_t \mid \mathcal{G}_s) = \mathbb{E}\left(\mathbb{E}(X_t \mid \mathcal{F}_s) \mid \mathcal{G}_s\right)$$

$$= \mathbb{E}(X_s \mid \mathcal{G}_s) = X_s$$

verificare che "qualcosa" valga per ogni termine di un σ-algebra $\mathcal{F}$

$\underline{A} \subset \mathcal{F}$

$$\mathbb{E}(X_t \mid \mathcal{F}_s)$$

$$\mathbb{P}(X_t \in I \mid \mathcal{F}_s) = \mathbb{P}(X_t \in I \mid X_s = x)$$

$$\mathbb{P}(X_t \in I \mid X_s = x, X_{s_1} = y_1, X_{s_2} = y_2 \dots X_{s_n} = y_n)$$
$$= \mathbb{P}(X_t \in I \mid X_s = x)$$

$\underline{\pi\text{ - sistema}}$ $\mathcal{P} \subset 2^{\Omega}$

$$A, B \in \mathcal{P} \implies A \cap B \in \mathcal{P}$$

λ - sistema $\mathcal{L}$

1. $A, B \in \mathcal{L}$ e $A \subset B \Rightarrow B \setminus A \in \mathcal{L}$

2. $A_1 \subset A_2 \subset A_3 \subset A_4 \Rightarrow \bigcup A_i \in \mathcal{L}$

$A_i \in \mathcal{L}$

Thm. $\mathcal{P} \subset \mathcal{L}$ e $\Omega \in \mathcal{P}$

allora $\mathcal{P} \subset \underbrace{\sigma(\mathcal{P})}_{\mathcal{F}} \subset \mathcal{L}$

$(\Omega, \mathcal{E}, \begin{smallmatrix}\mu_1 \\ \mu_2\end{smallmatrix})$

$\Omega \in \mathcal{P}$

μ_1

$\mu_1(F) = \mu_2(F) \quad \forall F \in \mathcal{P}$

$\mathcal{P}$ è π-sistema

μ_2

$\Rightarrow \mu_1(F) = \mu_2(F) \quad \forall F \in \sigma(\mathcal{P})$?

$\mathcal{L} =$ tutti gli insiemi $F^{\in \mathcal{E}}$: $\mu_1(F) = \mu_2(F)$

$\mathcal{L}$ è un λ-sistema $(\mathcal{P} \subset \mathcal{L})$

1. $A \subset B \quad A, B \in \mathcal{L}$ $\quad \mu_1(A) = \mu_2(A)$

$\mu_1(B) = \mu_2(B)$

$B \setminus A \in \mathcal{L}$

$\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) \qquad \mu_2(B \setminus A) = \mu_2(B) - \mu_2(A)$

2. $A_1 \subset A_2 \subset A_3 \subset \dots$

$\mu_1(A_1) = \mu_2(A_1)$
$\mu_1(A_2) = \mu_2(A_2)$

$$\begin{array}{ccc} \mu_1(A_k) & \uparrow & \mu_1(A) \\ \parallel & & \parallel \\ \mu_2(A_k) & \uparrow & \mu_2(A) \end{array}$$

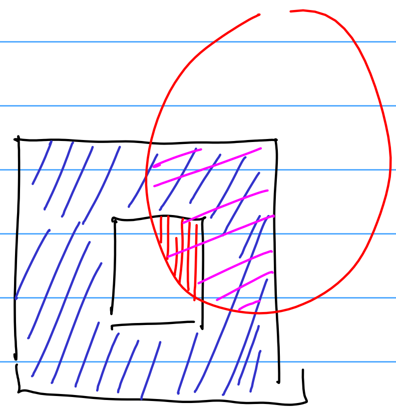
Dimostrazione. π-sistem $\mathcal{P}$ λ-sistem $\mathcal{L}$

tali che $\mathcal{P} \subset \mathcal{L}$ $\Omega \in \mathcal{P}$.

$\mathcal{P} \subset \mathcal{L}_1 = \{ E \in \mathcal{L} : \forall C \in \mathcal{P} \ E \cap C \in \mathcal{L} \} \subset \mathcal{L}$

$\boxed{\mathcal{L}_1 \text{ è } \lambda\text{-sistema}}$

$A \subset B \in \mathcal{L}_1$



$$(B \setminus A) \cap C = \underset{\underset{\mathcal{L}}{\uparrow}}{(B \cap C)} \setminus \underset{\underset{\mathcal{L}}{\uparrow}}{(A \cap C)} \in \mathcal{L}$$

$A \cap C \subset B \cap C$

$$\underset{\underset{\mathcal{L}_1}{\uparrow}}{A_1} \subset \underset{\underset{\mathcal{L}_1}{\uparrow}}{A_2} \subset \underset{\underset{\mathcal{L}_1}{\uparrow}}{A_3} \subset \dots \dots \subset A = \bigcup_k A_k$$

$$A \cap C = \bigcup_{i=1}^{\infty} (\underbrace{A_i \cap C}_{\in \mathcal{L}})$$

$$\mathcal{P} \subset \mathcal{L}_2 = \{ E \in \mathcal{L}_1 : \forall A \in \mathcal{L}_1,\ E \cap A \in \mathcal{L} \} \subset \mathcal{L}_1$$

$\mathcal{L}_2$ è un λ-sistema

hanno gettato le carte

$$\mathcal{P} \subset \underline{\mathcal{L}_2} \subset \underline{\mathcal{L}_1} \subset \underline{\mathcal{L}}$$

[$\tilde{\mathcal{L}}$ il più piccolo λ-sistema (che contiene $\mathcal{P}$) $\Omega \in \tilde{\mathcal{L}}$
tale che $\Omega \in \tilde{\mathcal{L}}$.

$$\mathcal{P} \subset \underbrace{\tilde{\mathcal{L}}_2 \equiv \tilde{\mathcal{L}}_1 \equiv \tilde{\mathcal{L}}}_{\text{è un } \pi\text{-sistema}} \subset \mathcal{L}$$

$$\forall A, B \in \tilde{\mathcal{L}} \Rightarrow A \cap B \in \tilde{\mathcal{L}}$$

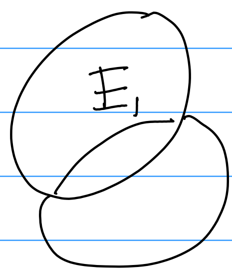
$$\tilde{\mathcal{L}}_2 = \{ E \in \tilde{\mathcal{L}}_1 : \forall A \in \tilde{\mathcal{L}}_1 : A \cap E \in \tilde{\mathcal{L}} \}$$

Se $\mathcal{L}$ λ-sistema e un π-sistema con $\Omega \in \mathcal{L} \Rightarrow$ $\mathcal{L}$ è una σ-algebra

$\Omega \in \mathcal{L}$.

$E \in \mathcal{L} \Rightarrow E^c \in \mathcal{L}$

$E_1, E_2 \dots \in \mathcal{L}$ $\quad \boxed{\bigcup E_i \in \mathcal{L}}$



$A_1 = E_1 \in \mathcal{L}$

$A_2 = E_1 \cup E_2$

$A_3 = E_1 \cup E_2 \cup E_3 = (E_1 \cup E_2) \cup E_3$

$A_1 \subset A_2 \subset A_3 \subset A_4 \dots$ $\quad \bigcup E_i = \bigcup A_i$

$E_1 \cup E_2 \in \mathcal{L}$?

$E_1 \in \mathcal{L} \quad E_2 \in \mathcal{L}$

$E_1^c \in \mathcal{L} \quad E_2^c \in \mathcal{L}$

$E_1^c \cap E_2^c \in \mathcal{L}$

$E_1 \cup E_2 = (E_1^c \cap E_2^c)^c \in \mathcal{L}$

$$\mathcal{P} \subset \sigma(\mathcal{P}) \subset \underset{\substack{\downarrow \\ \sigma\text{-algebra}}}{\tilde{\mathcal{L}}} \subset \mathcal{L}$$

Applicazione:

$$P\left(X_t \in I \mid \mathring{\mathcal{F}}_s\right)$$

$\forall F \in \mathring{\mathcal{F}}_s$

$$P\left((X_t \in I) \cap F\right) = \int_F P\left(X_t \in I \mid \mathring{\mathcal{F}}_s\right) dP$$

$$\mathcal{P} = \left\{ (X_{s_1} \in I_1) \cap (X_{s_2} \in I_2) \cap \dots \cap (X_{s_n} \in I_n) \right\}$$

al variare di n, $s_1 \leq s_2 \leq \dots \leq s_n \leq s$

e di $I_1, I_2, \dots, I_n \in \mathcal{B}$

$\mathcal{P} \subset \mathring{\mathcal{F}}_s \qquad \sigma(\mathcal{P}) \equiv \mathring{\mathcal{F}}_s$

$$F = \left(X_{s_1} \in I_1, X_{s_2} \in I_2, \dots, X_{s_n} \in I_n\right) \in \mathring{\mathcal{F}}_s$$

$$P\left((X_t \in I) \cap F\right) =$$

$$\int_{I_1 \times I_2 \times \dots \times I_n \times \mathbb{R}} P\left(X_t \in I \mid \overset{X_s = x}{X_{s_n} \cancel{= x_n}}, \cancel{X_{s_{n-1}} = x_{n-1}}, \cancel{\dots}, \cancel{X_{s_1} = x_1}\right) P\left(\underset{X_s \in dx}{X_{s_n} \in dx_n}, X_{s_{n-1}} \in dx_{n-1}, \dots, X_{s_1} \in dx_1\right)$$

$$P((X_t \in I) \cap F) = \int_F \underbrace{P(X_t \in I \mid (X_{s_1} \ldots X_{s_n})}_{\varphi(X_{s_1}, X_{s_2} \ldots X_{s_n})} \, dP$$

$$= \int_{(X_{s_1} X_{s_2} \ldots X_{s_n}) \in B} \varphi(X_{s_1} X_{s_2} \ldots X_{s_n}) \, dP = \int_B \varphi(x_1 x_2 \ldots x_n) \, P(X_{s_1} \in dx_1 \ldots X_{s_n} \in dx_n)$$

$$P((X_t \in I) \cap F) = \int_{I_1 \times I_2 \times \ldots \times I_n \times \mathbb{R}} \mathbb{P}(X_t \in I \mid X_s = x) \, P(X_s \in dx, X_{s_1} \in dx_1 \ldots X_{s_n} \in dx_n)$$

$\mu_1(F)$

$$\mu_2(F)$$

$$= \int_F \mathbb{P}(X_t \in I \mid X_s) \, dP$$

$$\int_F \mathbb{P}(X_t \in I \mid \mathcal{F}_s) \, dP = \int_F \mathbb{P}(X_t \in I \mid X_s) \, dP$$

$\forall F \in \mathcal{F}_s$