$\{X_t, \ t\in[0,T]\}$ $\qquad (\Omega, \mathcal{E}, \mathbb{P})$

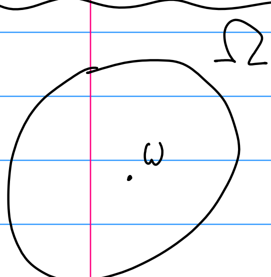
$[0,T] \longrightarrow [0,1]$

$$\left\{\begin{array}{l}
- \{\mu_t, \ t \in [0,T]\} \quad \mathbb{R} \\
- \{\mu_{t_1,t_2}, \ t_1 < t_2 \in [0,T]\} \quad \mathbb{R}^2 \\
- \{\mu_{t_1,t_2,t_3}, \ \dots\} \quad \mathbb{R}^3 \\
- \{\mu_{t_1 t_2 t_3 t_4}, \ \dots\dots\} \\
\quad \vdots
\end{array}\right\} \leftarrow \mathcal{M}_0$$

X_t

$X_{t_1} \quad X_{t_2}$

$X_{t_1}, \ X_{t_2} \ X_{t_3}$



$$\underbrace{t \longrightarrow X_t(\omega)}_{X_\cdot(\omega)}$$

ω fissato

traiettoria (cammino)

path

Lo spazio di tutte le funzioni $[0\,T] \longrightarrow \mathbb{R}$

spazio $= \mathbb{R}^{[0,T]}$

$$\underset{=}{\omega} \longrightarrow X_\cdot(\omega) \in \mathbb{R}^{[0,T]}$$

$$X : \Omega \longrightarrow \underline{\underline{\mathbb{R}^{[0,T]}}}$$

Teorema $\underbrace{X, \Omega}$ X', Ω'

$$\mu_{t_1 t_2 \dots t_n}(B) = \mathbb{P}\left(\omega \in \Omega : \left(X_{t_1}(\omega) \dots X_{t_n}(\omega)\right) \in B\right)$$

$$= \mathbb{P}'\left\{\omega' \in \Omega' : \left(X'_{t_1}(\omega') \dots X'_{t_n}(\omega')\right) \in B\right)$$

$$\mathbb{Q}_X \equiv \mathbb{Q}_{X'} \qquad (= Q)$$

teorema: $\left(\mathbb{R}^{[0,T]}, \sigma(\mathcal{C}), \mathbb{Q}\right)$ M6

$$Z_t(\omega) \triangleq \omega(t) \quad ||$$

$$\mathbb{Q}_Z \equiv Q$$

Teorema di Kolmogorov (di esistenza)

$$Q\left(C_{t_1 t_2 \dots t_n, B}\right) \stackrel{\wedge}{=} \mu_{t_1 t_2 \dots t_n}(B)$$

$\to$ riesce ad estendere Q da $\mathcal{C}$ alla σ-algebra generata di $\mathcal{C}$ $(\sigma(\mathcal{C}))$

$$\mu \longrightarrow Q$$

Supponiamo che tutte le $\mu_{\dots}$ siano gaussiane

$\mu_{t_1 t_2 \dots t_n}$ è gaussiana

$$\varphi_{t_1 t_2 \dots t_n}(S) = e^{i\langle m_{t_1 t_2 \dots t_n}, S\rangle - \frac{1}{2}\langle A_{t_1 t_2 \dots t_n} S, S\rangle}$$

$m(t)$, $\varphi(s,t)$ $\quad s,t \in [0,T]$

$$m(t) = 0 \qquad \varphi(t,s) = s \wedge t$$

$(\Omega, \mathcal{E}, \mathbb{P})$

$t_1\ t_2 \dots t_n \qquad \mu_{t_1 t_2 \dots t_n}$

$$\mu_{t_1 t_2 \dots t_n}(B) = \mathbb{P}\left((X_{t_1} X_{t_2} \dots X_n) \in B\right)$$

è una misura gaussiana

$$\boxed{\exp\left\{ i \langle m_{t_1 t_2 \dots t_n}, s\rangle - \frac{1}{2}\langle A_{t_1 t_2 \dots t_n} s, s\rangle \right\}}$$

$$\underline{m_{t_1 t_2 \dots t_n}} = \mathbb{E}\left((X_{t_1}, X_{t_2}, \dots, X_{t_n})\right) =$$

$$= \left(\mathbb{E}(X_{t_1}), \mathbb{E}(X_{t_2}), \dots, \mathbb{E}(X_{t_n})\right)$$

$$m(t) = \mathbb{E}(X_t) = (\underline{m}(t_1), \underline{m}(t_2), \dots, \underline{m}(t_n))$$

$$\| $$

$$\int_{\mathbb{R}} x \, \mu_t(dx)$$

$A_{t_1 t_2 \dots t_n}$ $\qquad (X_{t_1}, X_{t_2}, \dots, X_{t_n})$

$a_{ij} \qquad 1 \le i \le n \quad 1 \le j \le n$

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bullet & \end{pmatrix} \end{array}$$

$$a_{ij} = \mathbb{E}\left((X_{t_i} - m(t_i))(X_{t_j} - m(t_j)) \right)$$

$$= \int_{\mathbb{R}^2} (x_i - m(t_i))(x_j - m(t_j)) \, \mu_{t_i t_j}(dx_i dx_j)$$

$$\varphi(t,s) = \mathbb{E}\left((X_t - m(t))(X_s - m(s)) \right)$$

$$a_{ij} = \varphi(t_i, t_j)$$

$$\varphi(t,s) = \varphi(s,t)$$

$a_{ij} \qquad \sum_{i,j} a_{ij} z_i \bar{z}_j \ge 0$

Comunque scelgo n $\quad t_1\, t_2 \dots t_n \quad z_1 \dots z_n \in \mathbb{C}$

$$\sum_{i,j} \varphi(t_i, t_j)\, z_i \bar{z}_j \geq 0$$

$(\Omega, \mathcal{E}, P) \quad X_t : \Omega \to \mathbb{R}$

$P(X_t \dots X_{t_n})$ sono gaussiane

$$\boxed{m(t) \quad , \quad \varphi(s,t)}$$

Esempio 1. (moto browniano)

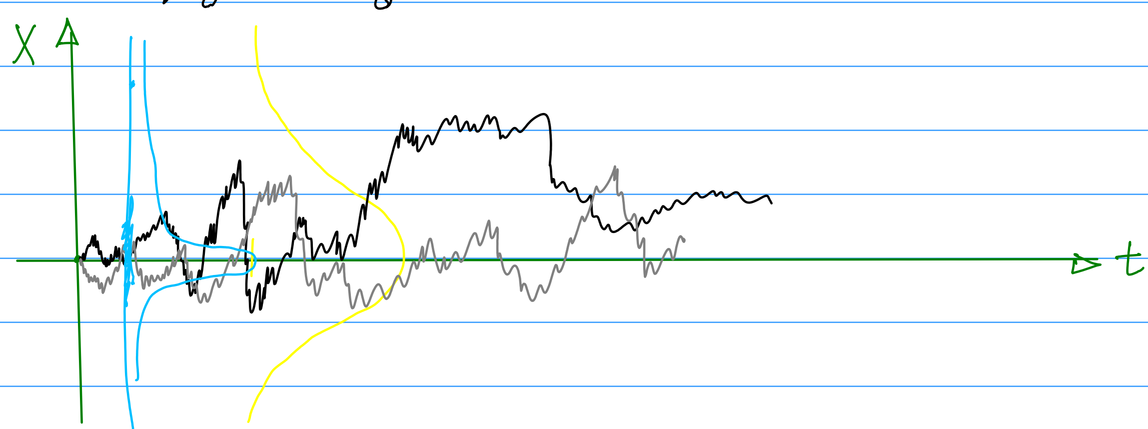
$$m(t) = 0 \qquad \varphi(s,t) = s \wedge t$$

$$X_t \sim N\left(m(t), \varphi(t,t)\right)$$

Esempio 1. $X_t \sim N(0, t)$ ha densità

$$\frac{1}{\sqrt{2\pi t}}\, e^{-\frac{x^2}{2t}}$$

$$X_0 \sim \delta_0$$



$$\varphi(s,t) = s \wedge t$$

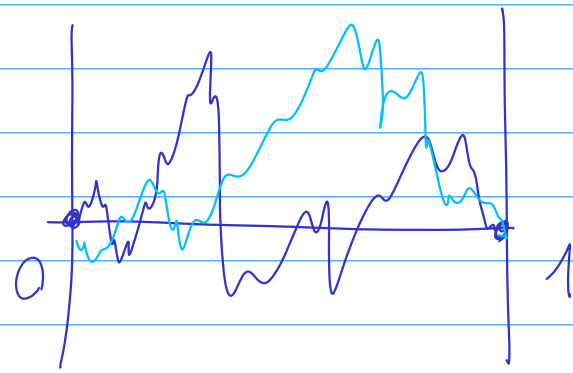
$$A_{t_1 t_2 \dots t_n} = \begin{pmatrix} t_1 & t_1 & t_1 & & t_1 \\ t_1 & t_2 & t_2 & & t_2 \\ \vdots & t_2 & t_3 & \dots & t_3 \\ \vdots & \vdots & \vdots & & \\ t_1 & t_2 & t_3 & & t_n \end{pmatrix}$$

$$\varphi(t_1, t_2) = \det \begin{vmatrix} t & t \\ t & s \end{vmatrix} = ts - t^2 = t(s-t)$$

$$\begin{pmatrix} t_1 & t_1 & & & t_1 \\ 0 & t_2-t_1 & t_2-t_1 & \dots & t_2-t_1 \\ \vdots & & & & \\ 0 & 0 & 0 \quad 0 & t_{n-2}-t_{n-1} & \\ 0 & 0 & 0 \quad 0 & t_{n-1}-t_{n-2} & t_{n-1}-t_{n-2} \\ 0 & 0 & 0 \quad 0 & \dots & t_n - t_{n-1} \end{pmatrix} = t_1 (t_2 - t_1)(t_3 - t_2) \dots\dots (t_n - t_{n-1})$$

Esempio 2. $\quad m(t) = 0 \qquad \varphi(s,t) = s \wedge t - st$

$0 \le s,t \le 1$



Ponte browniano $\qquad$ Brownian bridge

Moto browniano X_t

$$Y_t = X_t - t X_1 \qquad 0 \leq t \leq 1$$

1. Y_t è un processo gaussiano? SI

$$(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) = \begin{pmatrix} -t_1 & 1 & 0 & 0 & 0 & 0 \\ -t_2 & 0 & 1 & 0 & 0 & 0 \\ -t_3 & 0 & 0 & 1 & \cdots & 0 \\ -t_n & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_{t_1} \\ \vdots \\ X_{t_n} \end{pmatrix}$$

$$m(t) = \mathbb{E}(Y_t) = \mathbb{E}(X_t - t X_1)$$
$$= \mathbb{E}(X_t) - t\, \mathbb{E}(X_1) = 0$$

$$\varphi(s,t) = \mathbb{E}(Y_t Y_s) = \mathbb{E}\left((X_t - tX_1)(X_s - sX_1)\right)$$

$$= \mathbb{E}\left[\underline{X_t X_s} - \underline{t X_1 X_s} - s\, \underline{X_t X_1} + ts X_1^2\right]$$

$$= t \wedge s - ts - \cancel{st} + \cancel{ts}$$

Esempio 3. X_t moto browniano

$$Y_t = e^{-t} X_{e^t}$$

$\rightarrow$ esercizio $m(t) = 0$ $\varphi(t,s) = e^{-|t-s|}$

Y_t processo di Ornstein-Uhlenbeck

processo stazionario.

Esercizio X_t

$$Z_t = \frac{1}{\sqrt{c}} X_{ct}$$

$$(Z_{t_1}, Z_{t_2}, \dots, Z_{t_n}) = N \begin{pmatrix} X_{ct_1} \\ \vdots \\ X_{ct_n} \end{pmatrix}$$

$$\mathbb{E}(Z_t) = 0$$

$$\varphi(s,t) = \mathbb{E}(Z_s Z_t) = \mathbb{E}\left(\frac{1}{c} X_{ct} X_{cs}\right)$$

$$= \frac{1}{c} c\, t \wedge s = t \wedge s$$

X_{ct} $\quad c > 1 \quad c < 1$