Costruzione di proc. stoc.

processi gaussiani $\quad m(t) \quad \varphi(s,t)$

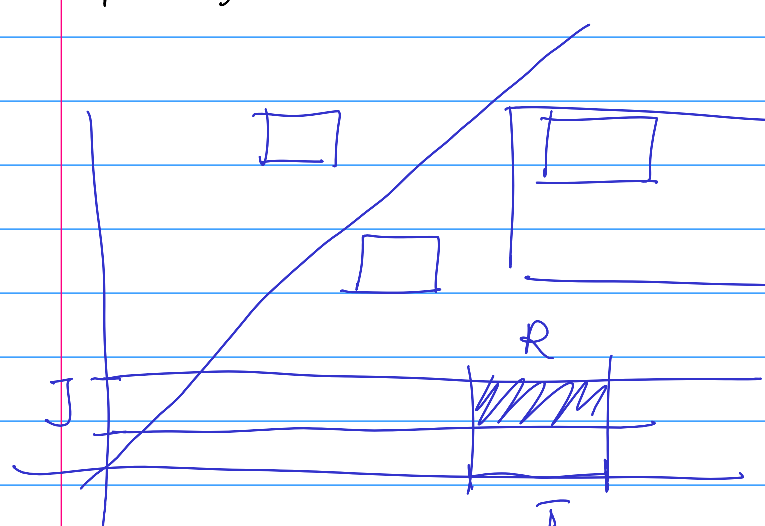
$$X_t = U \qquad t \geq 0 \qquad U$$

$$(\Omega, \mathcal{E}, \mathbb{P})$$

$$\omega : t \to X_t(\omega) = U(\omega)$$

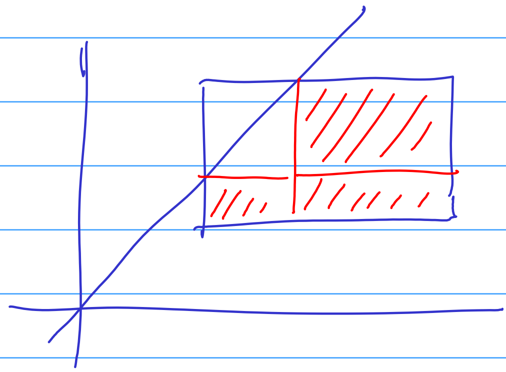
$$X : \Omega \longrightarrow \mathbb{R}^{[0,T]}$$

$$\mathcal{M} = \begin{cases} \mu_t = \mu \quad (\text{distr. di prob. di } U) \\ \mu_{t_1 t_2} = \\ \mu_{t_1 t_2 t_3} = \\ \mu_{t_1 t_2 t_3 t_4} = \end{cases}$$

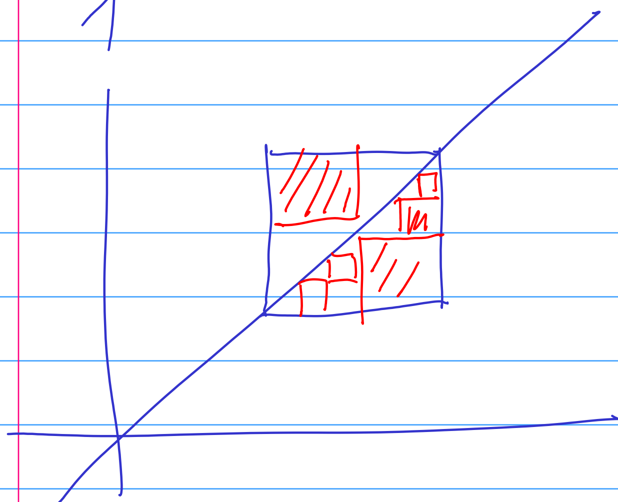


$$\mu_{t_1 t_2}(R) = \mathbb{P}(U \in I, U \in J)$$

$$\mu_{t_1 t_2}(\mathbb{R}\times\mathbb{R}) = 1$$

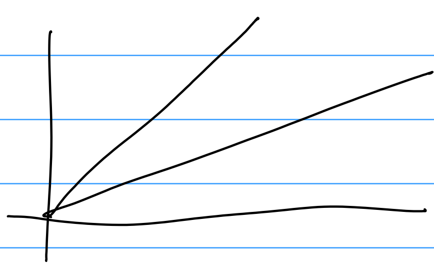


$$\mu_{t_1 t_2}(I \times I) = \mathbb{P}(U\in I, U\in I) = \mathbb{P}(U\in I) = \mu(I)$$



$\varphi(t)$ $\quad \varphi: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad (\mathbb{R}^+, [0,T])$

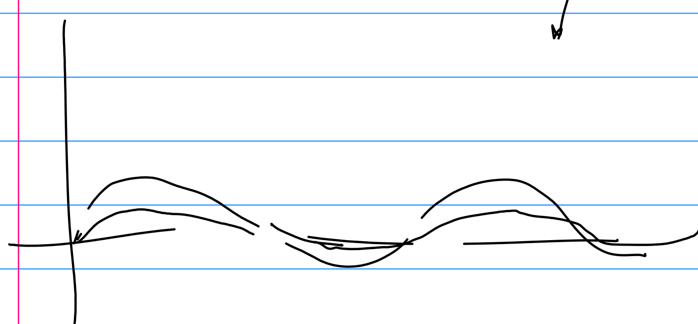
$$X_t = U\,\varphi(t)$$



$\varphi(t) = t$

$U(\omega) = 1$

$U(\omega') = 1/2$



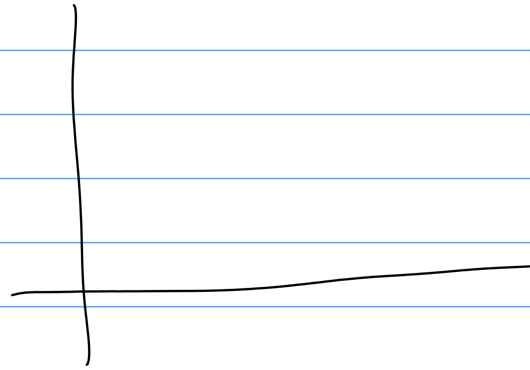
$\varphi(t) = \sin t$

$$\mu_t(I) = \mathbb{P}(X_t \in I) = \mathbb{P}(U\varphi(t) \in I)$$

$\varphi(t) = 0$ $\quad\quad$ $\varphi(t) \neq 0$ $\quad$ $U \in \tilde{I}_t = \{x : x\varphi(t) \in I\}$

δ_0

$\mu_{t_1 t_2}(I \times J)$



$$P(\varphi(t_1)U \in I, \varphi(t_2)U \in J)$$

$\varphi(t_1) \neq 0$
$\varphi(t_2) \neq 0$

$U \in \dfrac{I}{\varphi(t_1)}$ $\quad$ $U \in \dfrac{J}{\varphi(t_2)}$

$\dfrac{I}{\varphi(t_1)} \cap \dfrac{J}{\varphi(t_2)} = \emptyset$

$x\varphi(t_1) \in I$
$x\varphi(t_2) \in J$

Esercizio

$U_1, U_2, U_3, \dots, U_N$

$\varphi_1(t)\ \varphi_2(t)\ \varphi_3(t)\ \dots, \varphi_N(t)$

$$X_t = \sum_{i=1}^{N} U_i \varphi_i(t)$$

$U_1 \quad U_2 \quad U_3 \quad \dots \qquad \infty$

$\varphi_1(t) \quad \varphi_2(t) \quad \varphi_3(t) \quad \dots \qquad \infty$

$$X_t = \sum_{i=0}^{\infty} U_i \, \varphi_i(t)$$

$$N \longrightarrow \infty$$

$U_1 \quad U_2 \quad \dots \quad U_N$ <u>indipendenti</u>

<u>gaussiane</u> $N(0,1)$

$\varphi_1 \quad \varphi_2 \quad \dots \quad \varphi_N$ "<u>ortogonali</u>"

$[0,T]$

$$\int_0^T \varphi_i(t)\, \varphi_j(t)\, dt = 0 \qquad i \neq j$$

$L^2(0,T)$

$$\int_0^T \varphi_i(t)^2\, dt =$$

$$X_t = \sum_{i=1}^{N} \varphi_i(t)\, U_i$$

esercizio

X_t è un processo gaussiano

$m(t) = 0 \qquad \varphi(s,t)$

$$\varphi(s,t) = \mathbb{E}\left(X_s X_t\right) =$$

$$= \mathbb{E}\left(\sum_{i=1}^{N} \varphi_i(s) U_i \sum_{j=1}^{N} \varphi_j(t) U_j\right)$$

$$= \mathbb{E}\left(\sum_{i,j=1}^{N} \varphi_i(s)\varphi_j(t) U_i U_j\right) =$$

$$= \sum_{i,j=1}^{N} \varphi_i(s)\varphi_j(t)\, \mathbb{E}(U_i U_j) =$$

$$= \sum_{i=1}^{N} \varphi_i(s)\varphi_i(t) =: \varphi(s,t)$$

$$\int_0^T \varphi(s,t)\varphi_i(s)\,ds = \int_0^T \sum_{k=1}^{N} \varphi_k(s)\varphi_k(t) \cdot \varphi_i(s)\,ds$$

$$= \sum_{k=1}^{N} \varphi_k(t) \int_0^T \varphi_k(s)\varphi_i(s)\,ds = \varphi_i(t) \int_0^T \varphi_i(s)^2\,ds$$

Notazione $f(s)$

$$\Lambda f(t) = \int_0^T \varphi(s,t) f(s)\,ds$$

$$\Lambda \varphi_i(t) = \lambda_i \varphi_i(t)$$

$$\lambda_i = \int_0^T \varphi_i(s)^2 \, ds$$

$$\varphi(s,t) = s \wedge t \qquad ? \ \exists \lambda \longrightarrow f$$

$$\boxed{\int_0^T \underbrace{\varphi(s,t)}_{\text{nucleo}} f(s)\,ds = \lambda f(t)}$$

$$\int_0^T (s \wedge t) f(s)\,ds = \underline{\lambda f(t)}$$

misurabile

$$\int_0^T (s \wedge t) f(s)\,ds = \int_0^t s f(s)\,ds + t \int_t^T f(s)\,ds$$

$$\int_0^t s f(s)\,ds + t \int_t^T f(s)\,ds = \lambda f(t) \qquad (1)$$

$$\cancel{t f(t)} + \int_t^T f(s)\,ds - \cancel{t f(t)} = \lambda f'(t) \qquad (2)$$

$$(1) \Longrightarrow f(0) = 0 \qquad (2) \longrightarrow f'(T) = 0$$

dalla (2)

$$-f(t) = \lambda f''(t)$$

$\boxed{\lambda > 0}$ $\quad \lambda = 0 \quad \lambda < 0$

$$f'' + \frac{1}{\lambda} f = 0 \qquad z^2 + \frac{1}{\lambda} = 0$$

$$f(t) = A\cos\left(\frac{t}{\sqrt{\lambda}}\right) + B\sin\left(\frac{t}{\sqrt{\lambda}}\right)$$

$f(0)=0 \Rightarrow \quad f(t) = B\sin\left(\frac{t}{\sqrt{\lambda}}\right)$

$f'(T)=0 \quad f'(T) = B\frac{1}{\sqrt{\lambda}}\cos\left(\frac{T}{\sqrt{\lambda}}\right) = 0$

$$\cos\left(\frac{T}{\sqrt{\lambda}}\right)=0 \qquad \frac{T}{\sqrt{\lambda}} = (2k-1)\frac{\pi}{2}$$

$$\sqrt{\lambda} = \frac{T}{(2k-1)\pi/2} \qquad k = 1, 2, \dots$$

$$\lambda_k = \frac{T^2}{(2k-1)^2\pi^2/4} \qquad k = 1, 2, \dots$$

$$f_k(t) = B_k \sin\left((2k-1)\frac{\pi}{2}\frac{t}{T}\right)$$

$$X_t = \sum_{k=1}^{\infty} U_k \underbrace{B_k \sin\left(\frac{(2k-1)\pi}{2T} t\right)}_{\varphi_k(t)}$$

$$\int_0^T \varphi_k(t)^2 \, dt = \lambda_k = B_k^2 \int_0^T \sin^2\left(\frac{(2k-1)\pi}{2T} t\right) dt$$

$$\int_0^T \sin^2\left(\frac{(2k-1)\pi}{2T} t\right) dt$$

$$= \int_0^T \frac{1 - \cos\left(\frac{(2k-1)\pi}{T} t\right)}{2} \, dt$$

$$\frac{T}{2} - \frac{1}{2} \frac{\left[\sin \frac{(2k-1)\pi}{T} t\right]_0^T}{\frac{(2k-1)\pi}{T}} \equiv \frac{T}{2}$$

$$\frac{T^2}{(2k-1)^2 \pi^2/4} = B_k^2 \, T/2$$

$$\frac{T}{(2k-1)^2 \frac{\pi^2}{8}} = B_k^2 \qquad B_k = \frac{\sqrt{2T}}{(2k-1)\frac{\pi}{2}}$$

$$X_t = \frac{\sqrt{2T}}{\pi/2} \sum_{k=1}^{\infty} U_k \frac{\sin\left(\frac{(2k-1)\pi t}{2T}\right)}{2k-1}$$

Esercizio $\qquad \varphi(s,t) = s \wedge t - st$

$T = 1$

$\Omega = [0,1] \qquad \mathcal{B}_{[0,1]} \qquad$ Lebesgue

$\omega \in \Omega$

$$\omega = \sum_{k=1}^{\infty} \alpha_k \frac{1}{2^k}$$

$\alpha_1 \quad \alpha_2 \quad \dots \quad \dots$

$\alpha_1\ \alpha_3\ \alpha_6\ \alpha_{10}$ $\longrightarrow$ $X_1 = \sum_{k=1}^{\infty} \frac{\alpha_{j_k}}{2^k}$

$\alpha_2\ \alpha_5\ \alpha_9$

$\alpha_4\ \alpha_8$ X_2

α_7

$$\phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy \qquad U_k = \phi^{-1}(X_k)$$

$[0,T]$

$$Z_t = X_t - t X_1$$

$$Z_t = X_t - \frac{t}{T} X_T$$

Esercizio $\quad m(t) = 0 \qquad \varphi(s,t) = s \wedge t - \frac{st}{T}$

$$\int_0^T \varphi(s,t) f(s)\, ds = \lambda f(t)$$

$$\lambda = \int_0^T f(s)^2 ds$$

dopo

$$\int_0^t s f(s)\, ds + t \int_t^T f(s)\, ds - \frac{t}{T}\int_0^T s f(s)\, ds = \lambda f(t)$$

$$\cancel{t f(t)} + \int_t^T f(s)\, ds - \cancel{t f(t)} - \frac{1}{T}\int_0^T s f(s) ds = \lambda f'(t)$$

$$\begin{cases} -f(t) = \lambda f''(t) \\ f(0) = 0 \\ f(T) = 0 \end{cases} \qquad \lambda > 0$$

$$f(t) = \cancel{A\cos\left(\frac{t}{\sqrt{\lambda}}\right)} + B\sin\left(\frac{t}{\sqrt{\lambda}}\right)$$

$f(0)=0$

$$\sin\left(\frac{T}{\sqrt{\lambda}}\right) = 0 \qquad \frac{T}{\sqrt{\lambda}} = k\pi \quad k=1,2,3.\ldots$$

$$\boxed{\lambda_k = \frac{T^2}{k^2\pi^2}}$$

$$f_k(t) = B_k \sin\left(\frac{k\pi}{T}t\right)$$

$$B_k^2 \int_0^T \sin^2\left(\frac{k\pi}{T}t\right) dt = \frac{T^2}{k^2\pi^2}$$

$$B_k^2 \frac{T}{2} = \frac{T^2}{k^2\pi^2} \qquad B_k = \frac{\sqrt{2T}}{k\pi} \quad k=1,2,\ldots$$

$$Z_t = \frac{\sqrt{2T}}{\pi} \sum_{k=1}^{\infty} U_k \frac{\sin\left(\frac{k\pi}{T}t\right)}{k} \quad \ast$$

$$\varphi(s,t) = s\wedge t - \frac{st}{T} = \sum_{k=1}^{\infty} B_k^2 \sin\left(\frac{k\pi}{T}s\right)\sin\left(\frac{k\pi}{T}t\right)$$

$$= \frac{2T}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin\left(\frac{k\pi}{T} s\right) \sin\left(\frac{k\pi}{T} t\right)}{k^2}$$

$$0 \le s,t \le T$$

Wiener

$$s \wedge t = \frac{8T}{\pi^2} \sum_{k=1}^{\infty} \frac{\sin\left(\frac{(2k-1)\pi}{2T} s\right) \sin\left(\frac{(2k-1)\pi}{2T} t\right)}{(2k-1)^2}$$

$$0 \le s,t \le T$$

1. La serie converge (uniformemente) | da dimostrare

Le due serie convergono (unif.)

Riflessione X_t di Wiener $\Omega = \mathbb{R}^{[0T]}$

$\Omega = [0,1]$

$$Z_t = X_t - \frac{t}{T} X_T$$

Z_t è gaussiano, X_T entrambe a media 0

Z_t X_T sono indipendenti ?

$$\mathbb{E}\left(Z_t X_T\right) = \mathbb{E}\left(\left(X_t - \frac{t}{T} X_T\right) X_T\right) =$$

$$= \mathbb{E}\left(X_t X_T\right) - \frac{t}{T}\,\mathbb{E}\left(X_T^2\right)$$

$$t - \frac{t}{T}\cdot T = 0$$

$$X_t = \frac{t}{T} X_T + Z_t$$

$$= \frac{t}{\sqrt{T}} U_0 + \frac{\sqrt{2T}}{\pi} \sum_{k=1}^{\infty} U_k \frac{\sin\left(\frac{k\pi}{T} t\right)}{k}$$

$$0 \le t \le T$$

$$X'_t = \frac{1}{\sqrt{c}} X_{ct}$$

$$X''_t = t\, X_{1/t}$$

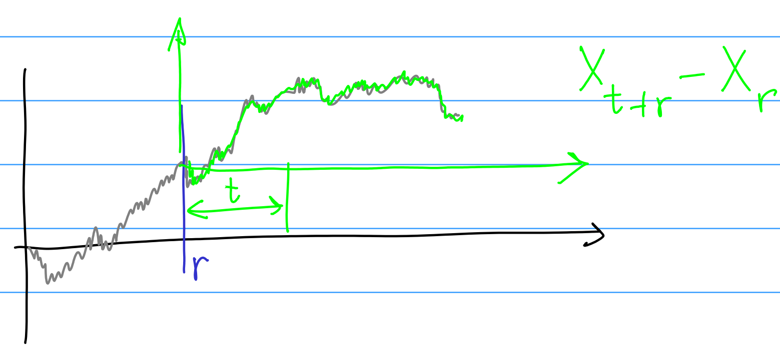
$$\varphi(s,t) = \mathbb{E}\left(X''_s X''_t\right)$$

$$= ts\, \mathbb{E}\left(X_{1/s} X_{1/t}\right)$$

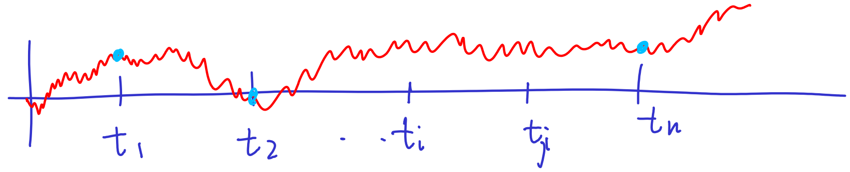
$$= ts\left(\frac{1}{s} \wedge \frac{1}{t}\right) = s \wedge t$$

$$\mathbb{E}\left({X''_t}^2\right) = t$$

$$X'''_t = X_{t+r} - X_r$$



<u>Wiener</u>



X_t

$$X_{t_1}, \quad X_{t_2} - X_{t_1}, \quad X_{t_3} - X_{t_2} \; \cdots\cdots, \quad X_{t_n} - X_{t_{n-1}}$$

$$Y_1 \qquad Y_2 \qquad Y_3 \qquad\qquad Y_n$$

$$P(Y_1 \in I_1, Y_2 \in I_2, \ldots, Y_n \in I_n) = P(Y_1 \in I_1) \cdots\cdots P(Y_n \in I_n)$$

$$\mathbb{E}(Y_i Y_j) = \qquad\qquad t_i < t_j$$

$$= \mathbb{E}\Big((X_{t_i} - X_{t_{i-1}})(X_{t_j} - X_{t_{j-1}}) \Big)$$

$$= \mathbb{E}\Big(X_{t_i} X_{t_j} - X_{t_{i-1}} X_{t_j} - X_{t_{j-1}} X_{t_i} + X_{t_{i-1}} X_{t_{j-1}} \Big)$$

$$\cancel{t_i} - \cancel{t_{i-1}} - \cancel{t_i} + \cancel{t_{i-1}} = 0$$

Corollario: Osserv. $X_{t+h} - X_t \sim N(0, h)$

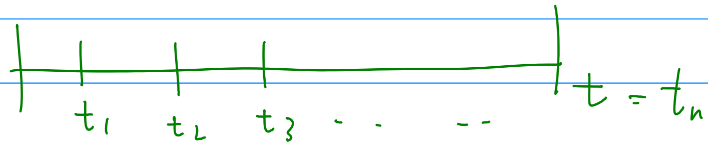
$$\mathbb{E}\big((X_{t+h} - X_t)^2\big) = h$$

$$X_{t_2} - X_{t_1} \sim N(0, t_2 - t_1)$$

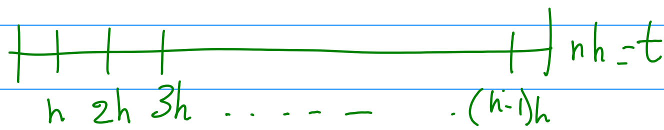
Osserv. $\dfrac{X_{t+h} - X_t}{\sqrt{h}} \sim N(0,1)$

$$X_t = X_{t_n} = X_{t_n} - X_{t_{n-1}} + X_{t_{n-1}} - X_{t_{n-2}} + X_{t_{n-2}} - X_{t_{n-3}} +$$

$$\cdots \underbrace{X_{t_1} - X_0}_{X_{t_1}} + \underbrace{X_0}_{=\,0}$$



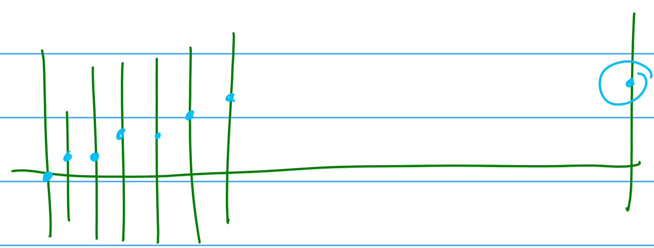
$$h = \frac{t}{n}$$

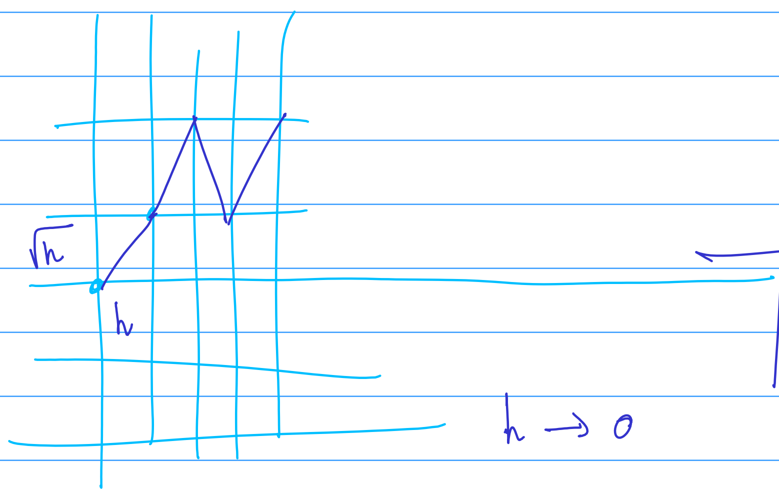


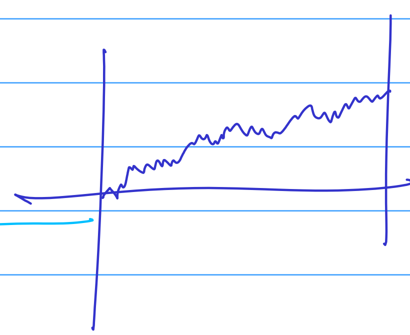
$$X_{t_i} - X_{t_{i-1}} = X_{ih} - X_{(i-1)h} \sim N(0, h)$$

$$Z_i = \frac{X_{t_i} - X_{t_{i-1}}}{\sqrt{h}} \sim N(0,1)$$

$$X_t = \sqrt{h}\left(Z_1 + Z_2 + \cdots + Z_n\right)$$







$h \to 0$

$$\left(\mathbb{R}^{[0,T]},\ \sigma(\mathcal{C}),\ W_0\right)$$

$$\left(C([0,T],\mathbb{R}),\ \sigma(\mathcal{C}),\ \underline{W}\right)$$

$\sigma(\mathcal{C}) \equiv \mathcal{B}$

misura di Wiener

— 1923 —

$D([0,T],\mathbb{R})$ cadlag