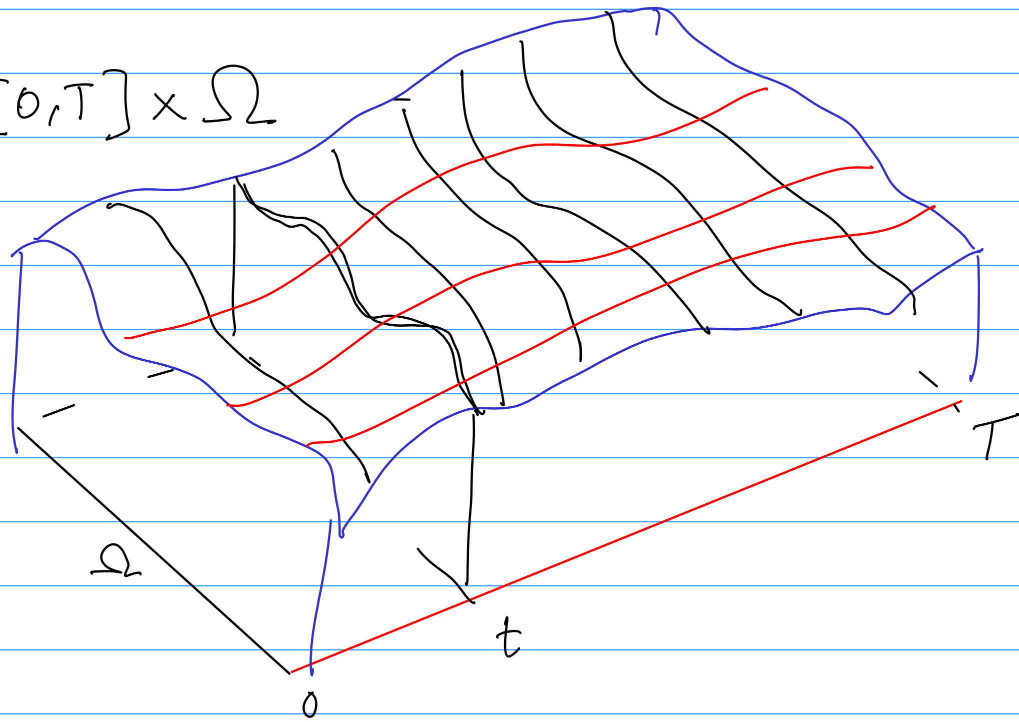
$(\Omega, \mathcal{E}, P)$ $\quad X_t : \Omega \longrightarrow \mathbb{R}^d$

$X_t(\omega)$ $\quad t \in [0,T]$

$X : [0,T] \times \Omega$



$\{X_t\} \longrightarrow \mathcal{M}$ (compatibile)

$\mathcal{M}$ è famiglia di misure gaussiane

$m(t)$, $\varphi(s,t)$

Probabilità condizionata:

$$\mathbb{P}\left((X_{t_1}, X_{t_2}, \dots X_{t_n}) \in B \right)$$

$$= \mu_{t_1 t_2 \dots t_n}(B)$$

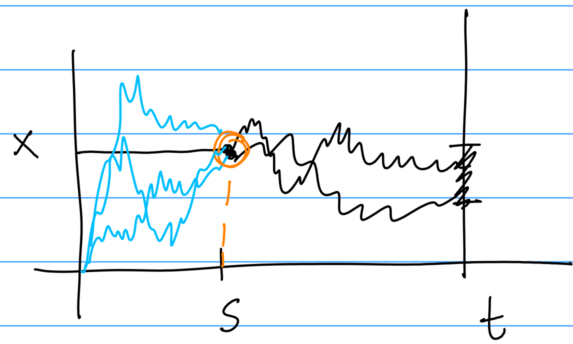
$$\mathbb{P}\left((X_{t_1}, X_{t_2} \dots X_{t_n}) \in B \;\middle|\; \underbrace{X_s = x} \right)$$

---

euristico $\qquad s < t$

$$\mathbb{P}\left(X_t \in I \mid X_s = x \right)$$

probabilità di transizione



$$\mu_t(B) = \mathbb{P}(X_t \in B)$$

$$= \int_{\mathbb{R}} \mathbb{P}\left(X_t \in B \mid X_s = x \right) \mathbb{P}\left(X_s \in dx \right)$$

$x \in \mathbb{R}$ $\qquad x \in \{x_1, x_2 \dots x_n\}$
$\qquad\qquad p_1 \; p_2 \; \dots \; p_n$

$$\mu_t(B) = \mathbb{P}(X_t \in B) = \sum_{i=1}^{n} \mathbb{P}(X_t \in B \mid X_s = x_i)\, \mathbb{P}(X_s = x_i)$$

$$\mathbb{P}(X_t \in B) = \mathbb{P}(X_t \in B,\ X_s \in \mathbb{R})$$
$$= \mathbb{P}(X_t \in B,\ X_s \in \{x_1 \dots x_n\}) =$$
$$= \mathbb{P}\Big(\bigcup_{i=1}^{n} (X_t \in B,\ X_s = x_i)\Big) =$$
$$= \sum_{i=1}^{n} \mathbb{P}(X_t \in B \mid X_s = x_i)\, \mathbb{P}(X_s = x_i)$$

$$\mu_t(B) = \int \mathbb{P}(X_t \in B \mid X_s = x)\, \mu_s(dx)$$

$$\mu_t(B) = \int \mathbb{P}(X_t \in B \mid X_0 = x)\, \underline{\underline{\mu_0(dx)}}$$

$$\left.\begin{array}{l} \boxed{\mu_0} \\ \mathbb{P}(X_t \in B \mid X_0 = x) \end{array}\right\} \longrightarrow \mu_t \qquad \underline{t > 0}$$

$$\mu_t(B) = \mathbb{P}(X_t \in B) = \mathbb{P}(X_t \in B,\ X_0 \in \mathbb{R}) =$$

$$\mu_{t_1,t_2}(B) = \mathbb{P}\left((X_{t_1}, X_{t_2}) \in B\right)$$

$$\mu_{t_1,t_2}(I_1 \times I_2) = \mathbb{P}\left(X_{t_1} \in I_1, X_{t_2} \in I_2\right) =$$

$$= \mathbb{P}\left(X_0 \in \mathbb{R}, X_{t_1} \in I_1, X_{t_2} \in I_2\right) =$$

$$= \int_{\mathbb{R}} \mathbb{P}\left(X_0 \in dx, X_{t_1} \in I_1, X_{t_2} \in I_2\right) =$$

$$= \int_{\mathbb{R} \times I_1} P\left(\underbrace{X_0 \in dx, X_{t_1} \in dy}, \underbrace{X_{t_2} \in I_2}\right) =$$

$$= \int_{\mathbb{R} \times I_1} \mathbb{P}\left(X_{t_2} \in I_2 \,\middle|\, X_0 = x, X_{t_1} = y\right) \mathbb{P}\left(X_0 \in dx, X_{t_1} \in dy\right)$$

$$= \int_{\mathbb{R} \times I_1} \underbrace{\mathbb{P}\left(X_{t_2} \in I_2 \,\middle|\, \cancel{X_0 = x,} X_{t_1} = y\right)}_{\mathbb{P}\left(X_{t_2} \in I_2 \,\middle|\, X_{t_1} = y\right)} \mathbb{P}\left(X_{t_1} \in dy \,\middle|\, X_0 = x\right) \mathbb{P}\left(X_0 \in dx\right)$$

μ_0

$$\underbrace{\mathbb{P}(X_t \in I \mid X_s = x)}$$

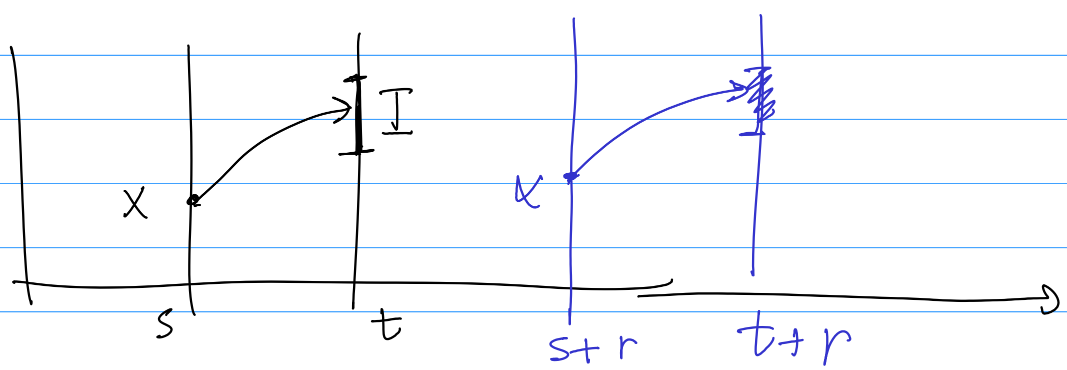
$$p(\underbrace{s, x}_{\text{variabili "backward"}} ; \underbrace{t, I}_{\text{variabili "forward"}})$$

$$(s, x, t) \longrightarrow p(s, x; t, \cdot)$$

è una misura su $\cdot$

$$p(s, x; t, I) = \varphi(t-s, x, I)$$

$$= p(t-s, x, I)$$



omogeneità nel tempo

$$p(t, x, I) = p_t(x, I)$$

$$p(s,x;t,I) = \int_I p(s,x;t,y)\,dy$$

$$p_t(x,I) = \int_I p_t(x,y)\,dy$$

$$p_t(x,y)$$

omogeneità spaziale $\quad p_t(x,y) = p_t(y-x)$

$p_t(x) \qquad p(s,x;t,I) = \int_I p_{t-s}(y-x)\,dy$

Relazione di <u>Chapman-Kolmogorov</u>

nel caso più semplice $\quad p_t(x)$

$$p_{t+s}(x) = \int_{\mathbb{R}} p_t(x-y)\,p_s(y)\,dy$$

$$= p_t * p_s$$

$$= \int_{\mathbb{R}} p_t(y)\,p_s(x-y)\,dy$$

Esempio

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

Esercizio

$$p_t * p_s = p_{t+s} \quad ?$$

$$\frac{1}{\sqrt{2\pi(t+s)}} e^{-\frac{x^2}{2(t+s)}} = \int_{\mathbb{R}} \frac{e^{-\frac{(x-y)^2}{2s}}}{\sqrt{2\pi s}} \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t}} dy$$

$$P(X_{t_1} \in I_1, \dots, X_{t_n} \in I_n) =$$

$$\int_{\mathbb{R}\times I_1 \times I_2 \times \dots \times I_n} P(X_{t_n} \in dx_n \mid \cancel{X_{t_1} = x_1, X_{t_2} = x_2, \dots,} X_{t_{n-1}} = x_{n-1})$$

$$P(X_{t_{n-1}} \in dx_{n-1} \mid \cancel{X_{t_1} = x_1 \dots} \; X_{t_{n-2}} = x_{n-2})$$

$$P(X_{t_{n-2}} \in dx_{n-2} \mid \cancel{X_{t_1} = x_1 \dots} \; X_{t_{n-3}} = x_{n-3})$$

$$P(X_{t_{n-3}} \in dx_{n-3} \mid \cancel{X_{t_1} = x_1 \dots} \; X_{t_{n-4}} = x_{n-4})$$

$$\vdots$$

$$P(X_{t_2} \in dx_2 \mid X_{t_1} = x_1) \; P(X_{t_1} \in dx_1 \mid X_0 = x_0)$$

$$P(X_0 \in dx_0)$$

prob. iniziale $P(X_0 \in dx_0)$ e le

prob. di transizione $\qquad s<t$

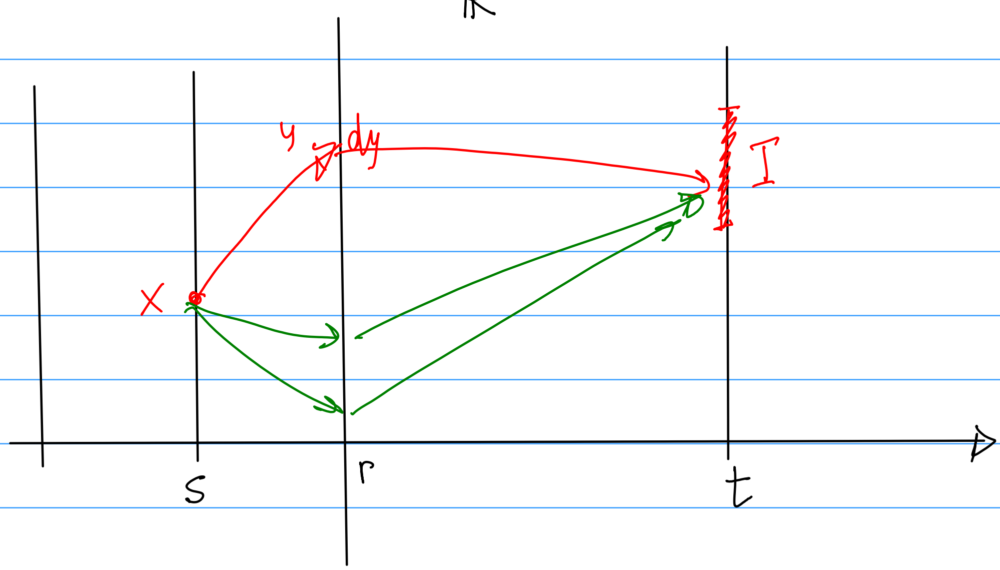
$$P(X_t \in dy \mid X_s = x) =: p(s,x;t,dy)$$

$$P(X_{t_1} \in I_1, X_{t_2} \in I_2, \dots, X_{t_n} \in I_n) =$$

$$\int_{\mathbb{R}\times I_1 \times \dots \times I_n} \mu_0(dx_0)\, p(0,x_0;t_1,dx_1)\, p(t_1,x_1;t_2,dx_2) \dots\dots p(t_{n-1},x_{n-1};t_n,dx_n)$$

Chapman-Kolmogorov $\qquad s<r<t$

$$p(s,x;t,I) = \int_{\mathbb{R}} p(s,x;r,dy)\, p(r,y;t,I)$$



$$(\Omega, \mathcal{E}, P) \qquad X_t \longrightarrow Q_x$$

$$\phi(\omega) \qquad \phi : \Omega \longrightarrow \mathbb{R} \quad \text{v.c.}$$

$$\mathbb{E}(\phi)$$

$$\phi(\omega) = F\left(X_{t_1}(\omega), X_{t_2}(\omega), \dots, X_{t_n}(\omega)\right)$$

$$\begin{tikzcd}
\Omega \arrow[r, "(X_{t_1} \dots X_{t_n})"] \arrow[dr, "\phi"'] & \mathbb{R}^n \arrow[d, "F"] \\
 & \mathbb{R}
\end{tikzcd}$$

$$\mathbb{E}(\phi) =$$

$$= \int_{\mathbb{R}^n} F(x_1 x_2 \dots x_n)\, P\left(X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n\right) =$$

$$= \int_{\mathbb{R} \times \mathbb{R}^n} F(x_1 \dots x_n)\, \mu_0(dx_0)\, p(0, x_0, t_1, dx_1) \dots p(t_{n-1}, x_{n-1}, t_n\, dx_n)$$

$$= \int_{\mathbb{R} \times \mathbb{R}^n} F(x_1 \dots x_n)\, \mu_0(dx_0)\, g_{t_1}(x_1 - x_0)\, g_{t_2 - t_1}(x_2 - x_1) \dots g_{t_n - t_{n-1}}(x_n - x_{n-1})\, dx_1 \dots dx_n$$

$$p(s, x; t, dy) = g_{t-s}(y - x)\, dy$$

Esempio: Processo di Wiener

$$m(t) = 0 \qquad \varphi(s,t) = s \wedge t$$

$$P(X_{t_1} \in I_1, \dots, X_{t_n} \in I_n) =$$

$$= \int_{I_1 \times \dots \times I_n} \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\det A}} e^{-\frac{1}{2}\langle A^{-1}x, x\rangle} dx$$

$$A = \begin{pmatrix} t_1 & \dots & \dots & t_1 \\ \vdots & t_2 & \dots & t_2 \\ \vdots & \vdots & \ddots & \\ \vdots & \vdots & & t_{n-1} \; t_{n-1} \\ t_1 & t_2 & & t_{n-1} \; t_n \end{pmatrix}$$

$$\det A = t_1 (t_2 - t_1)(t_3 - t_2)(t_4 - t_3) \dots (t_n - t_{n-1})$$

$$y = A^{-1} x \qquad Ay = x$$

$$\begin{cases} t_1 (y_1 + y_2 + \dots + y_n) = x_1 \\ t_1 y_1 + t_2 (y_2 + \dots + y_n) = x_2 \\ t_1 y_1 + t_2 y_2 + t_3 (y_3 + \dots + y_n) = x_3 \\ \vdots \\ t_{n-2} (y_{n-2} + y_{n-1} + y_n) = x_{n-2} \end{cases}$$

$$\begin{cases} t_1 y_1 + t_2 y_2 + \cdots + t_{n-1}(y_{n-1}+y_n) = x_{n-1} \\ t_1 y_1 + t_2 y_2 + \cdots + t_{n-1} y_{n-1} + t_n y_n = x_n \end{cases}$$

$$\begin{cases} \cancel{x_1}\underline{(y_1 + \cdots + y_n)} = x_1^2/t_1 = \dfrac{x_1 - 0}{t_1 - 0} \\ (x_2 - x_1)(y_2 + y_3 + \cdots + y_n) = \dfrac{(x_2 - x_1)^2}{t_2 - t_1} \\ \\ y_{n-2} + y_{n-1} + y_n = \dfrac{x_{n-2} - x_{n-3}}{t_{n-2} - t_{n-3}} \\ y_{n-1} + y_n = \dfrac{x_{n-1} - x_{n-2}}{t_{n-1} - t_{n-2}} \\ y_n = \dfrac{x_n - x_{n-1}}{t_n - t_{n-1}} \end{cases}$$

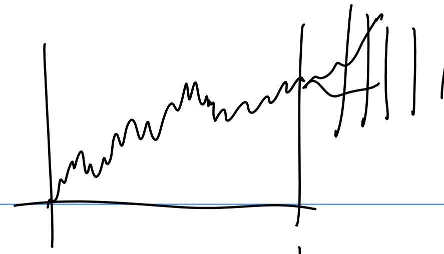
$$\langle A^{-1}x, x\rangle = \langle y, x\rangle = \underline{y_1 x_1} + y_2 x_2 + \cdots + y_n x_n$$

$$= \frac{x_1^2}{t_1} + \frac{(x_2 - x_1)^2}{t_2 - t_1} + \frac{(x_3 - x_2)^2}{t_3 - t_2} + \cdots + \frac{(x_n - x_{n-1})^2}{t_n - t_{n-1}}$$

$$P(X_{t_1} \in I_1 \cdots X_{t_n} \in I_n) =$$

$$= \int_{I_1 \times \cdots \times I_n} g_{t_1}(x_1)\, g_{t_2 - t_1}(x_2 - x_1)\, g_{t_3 - t_2}(x_3 - x_2) \cdots g_{t_n - t_{n-1}}(x_n - x_{n-1})\, dx_1 \cdots dx_n$$

$$\phi(\omega) = e^{\int_0^t X_s(\omega)\, ds}$$



$$\mathbb{E}(\phi) = \mathbb{E}\left(e^{\int_0^t X_s(\omega) ds}\right) = \mathbb{E}\left(\prod_0^t e^{X_s(\omega)\, ds}\right)$$

$$\int_0^t X_s(\omega)\, ds = \left[s X_s\right]_0^t - \int_0^t s\, dX_s = t X_t - \int_0^t s\, dX_s =$$

$$= t \int_0^t dX_s - \int_0^t s\, dX_s = \int_0^t (t-s)\, dX_s$$

$$= \mathbb{E}\left(e^{\int_0^t (t-s)\, dX_s(\omega)} \right) = \mathbb{E}\left(\prod_0^t e^{(t-s)\, dX_s(\omega)} \right) =$$

$$= \prod_0^t \mathbb{E}\left(e^{(t-s)\, dX_s} \right) =$$

$t - s = \alpha$

$dX_s \sim N(0, ds)$

$\beta = ds$

$$\mathbb{E}\left(e^{\alpha N_{0,\beta}} \right) = $$

$$= \int_{\mathbb{R}} e^{\alpha \xi} \frac{e^{-\frac{\xi^2}{2\beta}}}{\sqrt{2\pi\beta}} = e^{\frac{\alpha^2 \beta}{2}}$$

$$= \prod_{0}^{t} e^{\frac{1}{2}(t-s)^2 ds} = e^{\frac{1}{2}\int_0^t (t-s)^2 ds} = e^{t^3/6}$$

Il processo è a traiettorie continue !

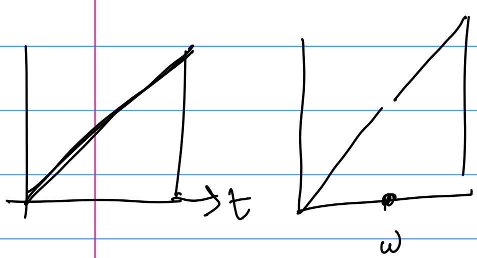
$\Omega = [0,1]$

$\mathcal{E} = \mathcal{B}$

$\mathbb{P} = \text{Lebesgue}$

$$X_t(\omega) = t \qquad t \in [0,1]$$

$$X'_t(\omega) = \begin{cases} t & \omega \neq t \\ 0 & \omega = t \end{cases}$$



esercizio : M è lo stesso