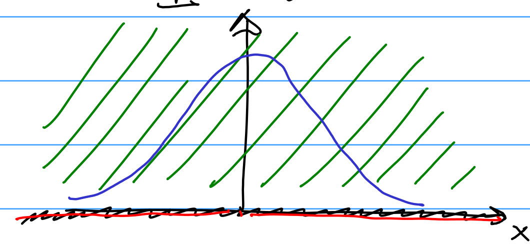
$p(s,x;t,I)$ $\qquad s < \quad < t$

$$p(s,x;t,I) = \int_{\mathbb{R}} p(s,x;r,dz)\, p(r,z;t,I)$$

$$g_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \qquad p(s,x;t,I) = \int_I g_{t-s}(y-x)\, dy$$

$$\frac{\partial g_t(x)}{\partial t} = \frac{1}{2} \frac{\partial^2 g_t(x)}{\partial x^2} \qquad x \in \mathbb{R}, \quad t \geq 0$$

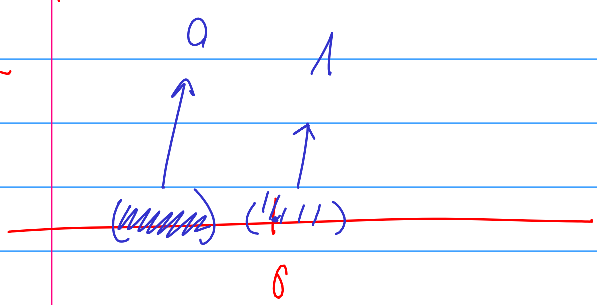


problema di Cauchy $\qquad g_0(x)$

– parabolico del secondo ordine

$$\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \varphi(x)\, dx \longrightarrow \varphi(0) \qquad \varphi \in C_c^\infty$$

$$= \int_{\mathbb{R}} \varphi(x)\, \delta_0(dx)$$



$a(x) \geq 0$

Più in generale
$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2} a(x) \dfrac{\partial^2 u}{\partial x^2} + b(x) \dfrac{\partial u}{\partial x} \\ u(0,x) = \delta_y(x) \end{cases}$$

soluzione fondamentale

$$u(t,x) = u(t,x,y) \geq 0$$

$$\int_{\mathbb{R}} u(t,x,y)\,dx = 1$$

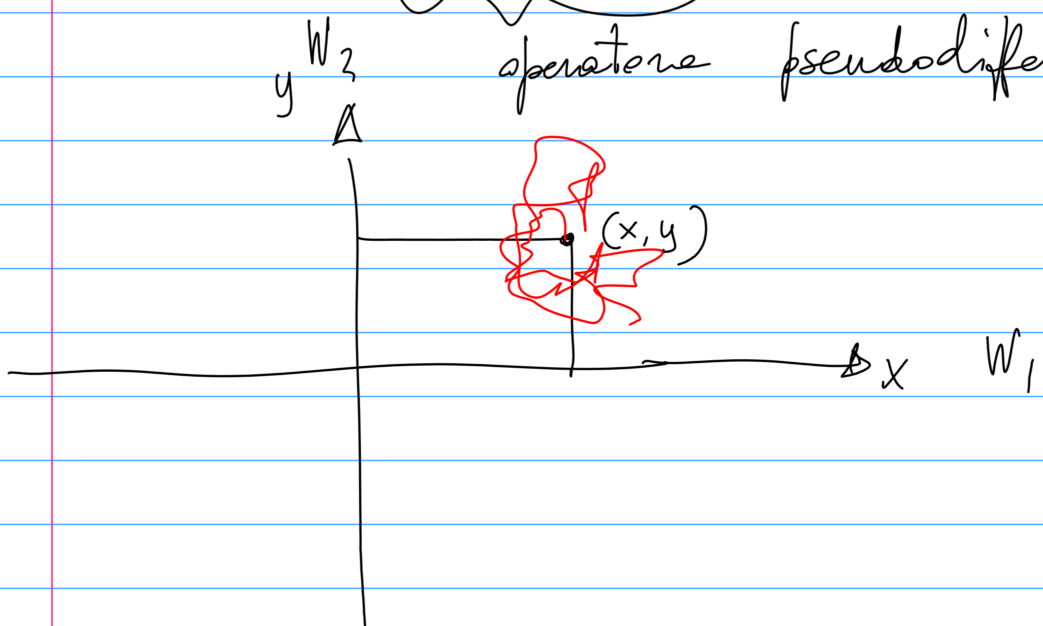
$$u(t,x,y) =: p_t(y,x)$$

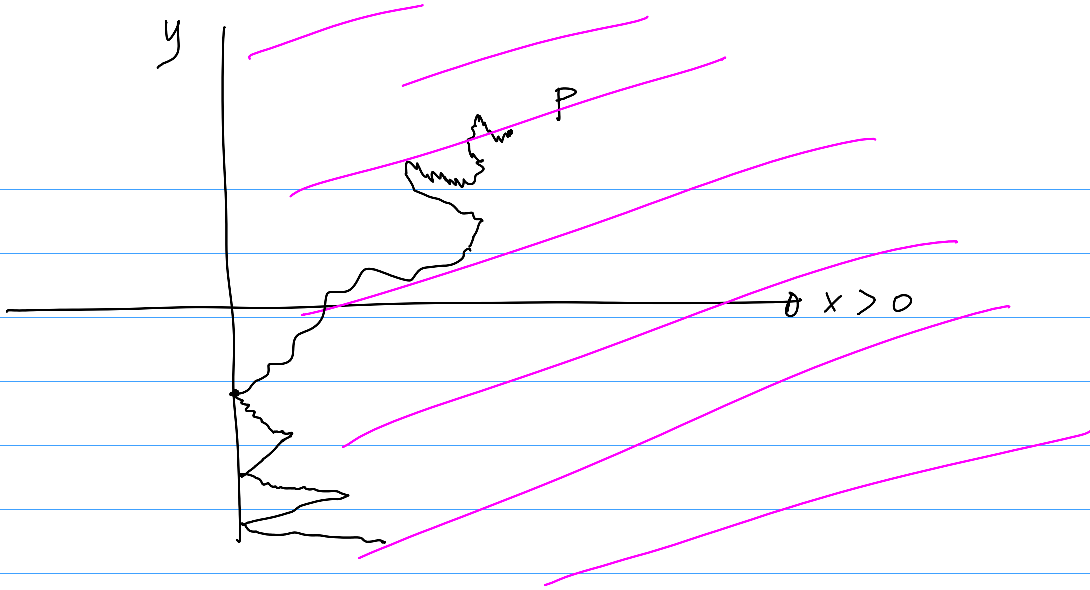
$$p_t(x) = \frac{1}{\pi}\,\frac{t}{t^2+x^2}$$

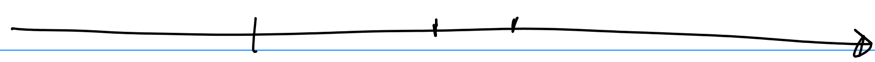
$$p_{t+s} = p_t * p_s$$

$$\frac{\partial u}{\partial t} = \underbrace{\sqrt{-\frac{\partial^2}{\partial x^2}}}_{\text{operatore pseudodifferenziale}}\, u$$

'60



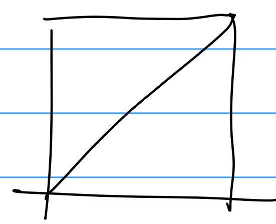




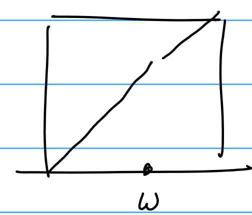
$\Omega = [0,1]$, $\mathcal{B}$, dx

$0 \leq t \leq 1$

$X_t(\omega) = t$



$$X'_t(\omega) = \begin{cases} t & t \neq \omega \\ 0 & t = 0 \end{cases}$$

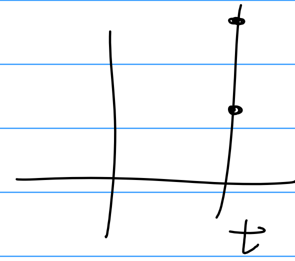


$$P\left((X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \in B \right) = P\left((X'_{t_1}, X'_{t_2} \ldots X'_{t_n}) \in B \right)$$

Due processi X_t, X_t' definiti su Ω

si dicono equivalenti

$$\forall t \qquad P(X_t \neq X'_t) = 0$$



$$A_t = (X_t \neq X'_t)$$

$$\boxed{\bigcup_t A_t = \Omega}$$

<u>Teorema</u> Se X e X' sono equivalenti allora $\mathcal{M} \equiv \mathcal{M}'$

<u>Dimostr.</u>

$$P\Big((X_{t_1}, X_{t_2} \dots X_{t_n}) \in B \Big) =$$

$$= P\Big(\cancel{(X_{t_1}, X_{t_2} \dots X_{t_n}) \in B}, \big[(X_{t_1} \neq X'_{t_1}) \cup (X_{t_2} \neq X'_{t_2}) \cup (X_{t_n} \neq X'_{t_n}) \big] \Big)$$

$$+ P\Big((X_{t_1}, X_{t_2} \dots X_{t_n}) \in B, \bigcap (X_{t_i} = X'_{t_i}) \Big)$$

$$\leq P\Big((X_{t_1} \neq X'_{t_1}) \cup \dots \cup (X_{t_n} \neq X'_{t}) \Big]$$

$$\leq \sum_{i=1}^{n} P(X_{t_i} \neq X'_{t_i}) = 0$$

$$= P\left((X'_{t_1}, X'_{t_2}, \dots, X'_{t_n}) \in B, \bigcap_i (X_t = X'_{t_i}) \right)$$

$$= P\left((X'_{t} \dots X'_{t_1}) \in B \right) \quad ∎$$

Teorema (Kolmogorov) | 1903 – 1986 |

Dato un processo X_t $t \in [0,T]$ $(\Omega, \mathcal{E}, \mathbb{P})$

tale che

$$\mathbb{E}\left(|X_t - X_s|^{\alpha} \right) \leq C \, |t-s|^{1+\beta}$$

$\alpha, \beta > 0$

allora esiste un processo X'_t, equivalente a X_t, le cui traiettorie sono Hölderiane di ordine $\gamma < \frac{\beta}{\alpha}$.

Caso del processo di Wiener

$$\mathbb{E}\left(|X_t - X_s|^{2m} \right) = c_m \, |t-s|^{m}$$

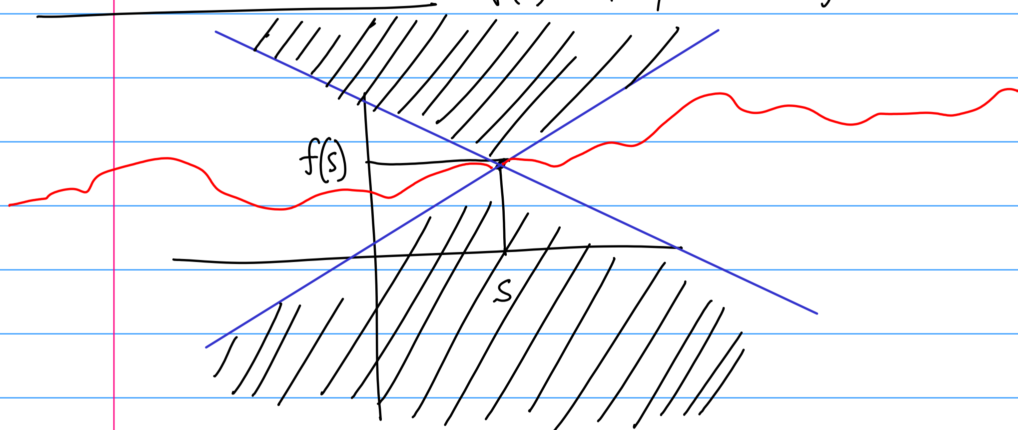
$\beta = m-1 \qquad \alpha = 2m \qquad \gamma < \frac{m-1}{2m} = \frac{1}{2} - \frac{1}{2m}$

Per il processo di Wiener $\gamma < \frac{1}{2}$

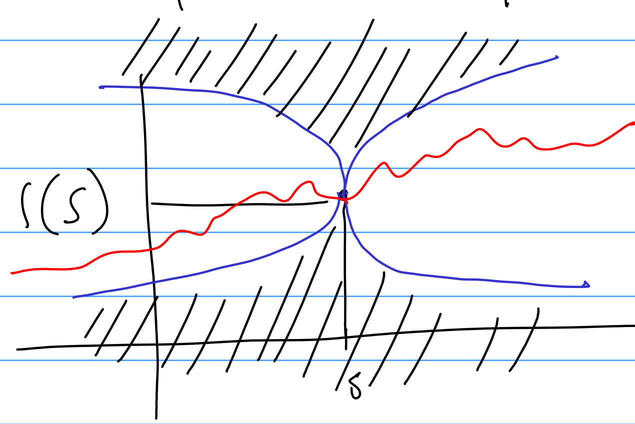
Hölderianità Hölder 0. $[0,T]$

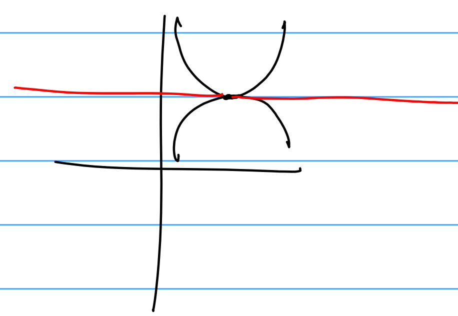
Lipschitzianità <u>Lipschitz</u> $|f(t) - f(s)| \leq C|t-s|$

<u>Derivata limitata</u> $f(s) - C|t-s| \leq f(t) \leq f(s) + C|t-s|$



γ $$|f(t) - f(s)| \leq C|t-s|^{\gamma}$$





~~Teorema di~~ <u>Dimostrazione.</u>