Teorema di Kolmogorov $\quad (\Omega, \mathcal{E}, P) \quad X_t \quad [0,T]$

(1) $$\mathbb{E}\left(|X_t - X_s|^{\alpha}\right) \leq C\, |t-s|^{1+\beta}$$

$\Rightarrow \exists\, X'_t \ (\Omega, \mathcal{E}, P)$, equivalente a X_t, con traiettorie continue_ [sono hölderiane di $\gamma < \frac{\beta}{\alpha}$

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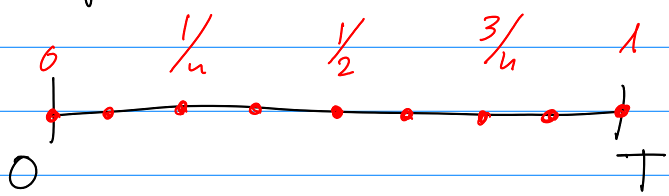
1. scelta dei punti



$$D_m = \left\{ \frac{k}{2^m},\ k = 0, 1, 2, \dots, 2^n \right\}$$

$$D = \bigcup_{m=0}^{\infty} D_m$$

2° studiamo cosa succede su $D_m \qquad t_i \in D_m$

$t_0, t_1, t_2, \dots, t_{2^m}$

$$\mathbb{E}\, |X_{t_{i+1}} - X_{t_i}|^{\alpha} \leq C\, (t_{i+1} - t_i)^{1+\beta}$$

$$= C \frac{1}{2^{m(1+\beta)}}$$

Chebyshev $\quad \mathbb{P}\left(|X_{t_{i+1}} - X_{t_i}| > \lambda\right) \leq \frac{1}{\lambda^{\alpha}} \mathbb{E}\left(|X_{t_{i+1}} - X_{t_i}|^{\alpha}\right)$

$$\mathbb{P}\left(|X_{t_{i+1}} - X_{t_i}| > \lambda \right) \leq \frac{C}{\lambda^\alpha} \frac{1}{2^{m(1+\beta)}}$$

$$\mathbb{P}\left(\max_i |X_{t_{i+1}} - X_{t_i}| > \lambda \right) =$$

$$= \mathbb{P}\left[\bigcup_i \left(|X_{t_{i+1}} - X_{t_i}| > \lambda \right) \right]$$

$$\leq \sum_{i=0}^{2^m - 1} \mathbb{P}\left(|X_{t_{i+1}} - X_{t_i}| > \lambda \right)$$

$$\leq 2^m \frac{C}{\lambda^\alpha} \frac{1}{2^{m(1+\beta)}} = \frac{C}{\lambda^\alpha 2^{m\beta}}$$

3 passo

$$\lambda = \frac{1}{2^{m\gamma}}$$

$$\boxed{\mathbb{P}\left(\max_i |X_{t_{i+1}} - X_{t_i}| > 2^{-m\gamma} \right) \leq \frac{C}{2^{m(\beta - \alpha\gamma)}}}$$

$$\beta - \alpha\gamma > 0 \qquad \gamma < \frac{\beta}{\alpha}$$

$$\overbrace{\text{Lemma di Borel - Cantelli}}$$

Sia A_j una successione di eventi $\subset \Omega$

con la proprietà $\sum_{j=1}^{\infty} \mathbb{P}(A_j) < \infty$

allora $\mathbb{P}(A) = 0$.

$$\sum_{j=k}^{\infty} \mathbb{P}(A_j) \longrightarrow 0 \quad k \to \infty$$

$$\mathbb{P}\left(\bigcup_{j=k}^{\infty} A_j\right) \leq \sum_{j=k}^{\infty} \mathbb{P}(A_j) \longrightarrow 0$$

$$A = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} A_j \subset \bigcup_{j=k}^{\infty} A_j$$

$$\mathbb{P}(A) \leq \mathbb{P}\left(\bigcup_{j=k}^{\infty} A_j\right) \xrightarrow[k \to \infty]{} 0$$

$$\Rightarrow \mathbb{P}(A) = 0$$

4 passo.

$$A_m = \left(\max_i |X_{t_{i+1}} - X_{t_i}| > 2^{-m\gamma}\right)$$

$$\sum_{m=0}^{\infty} \mathbb{P}(A_m) \leq C \sum_{m=0}^{\infty} \frac{1}{2^{m(\beta - \alpha\gamma)}} < +\infty \qquad \beta - \gamma\alpha > 0$$

$$A = \bigcap_{k=0}^{\infty} \bigcup_{m=k}^{\infty} A_m \qquad \boxed{\mathbb{P}(A) = 0}$$

⑤ passo

$$\omega \notin A$$

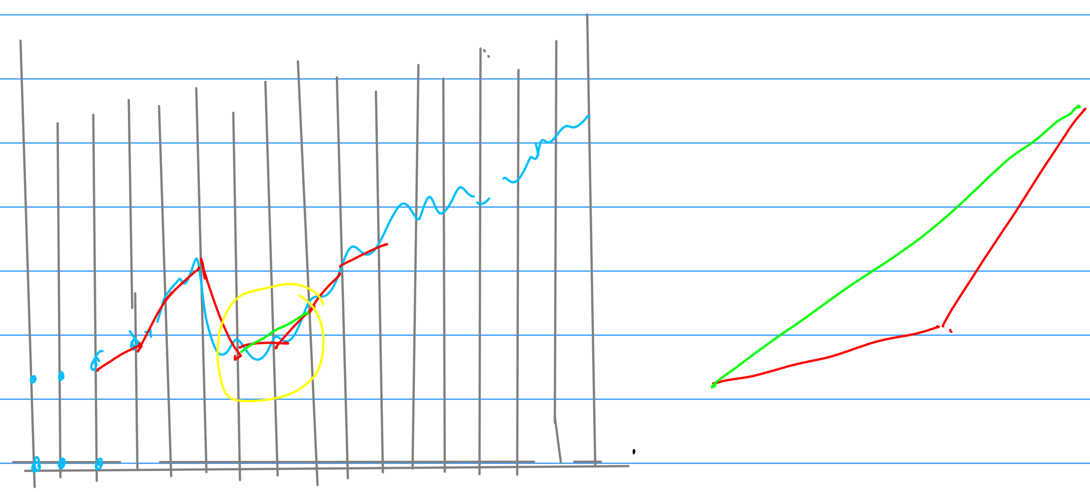
$$\omega \in \left[\bigcap_{k=0}^{\infty} \bigcup_{m=k}^{\infty} A_m\right]^c = \bigcup_{k=0}^{\infty} \bigcap_{m=k}^{\infty} A_m^c$$

$$\exists\, \bar{k}(\omega) \;:\; m \geq \bar{k} \quad \underbrace{\omega \in A_m^c}$$

$$\Updownarrow$$

$$|X_{t_{i+1}}(\omega) - X_{t_i}(\omega)| \leq 2^{-m\gamma} \qquad \forall i = 0, \dots, 2^m - 1$$

II parte.



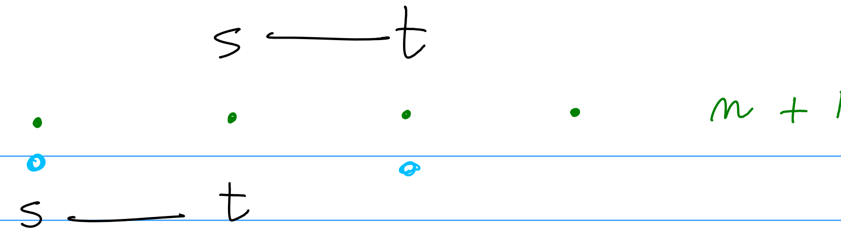
$$|t - s| < \delta \qquad t, s \in D$$

$$t, s \in D_m$$

$$\boxed{|t - s| < \delta = \frac{1}{2^m}}$$

(n fisso)

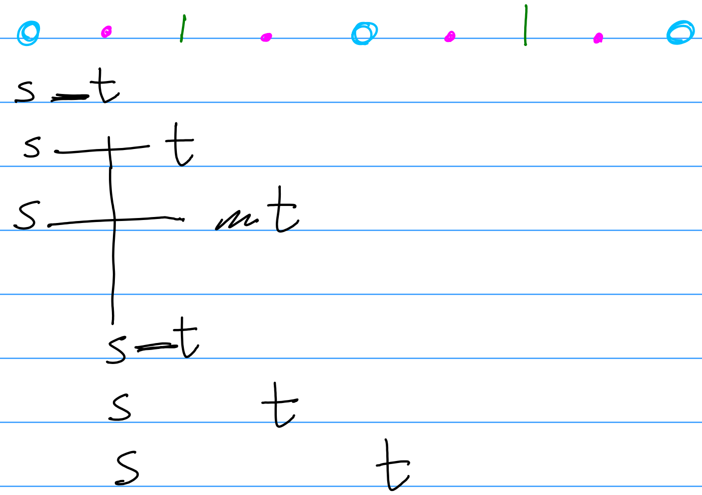
$m = n+1$



$s, t \in D_m$

$$|X_t - X_s| \le \frac{1}{2^{(n+1)\gamma}}$$

$m = n+2$



$$\frac{1}{2^{(n+2)\gamma}}$$

$$|X_t - X_s| \le \frac{1}{2^{(n+1)\gamma}} + \frac{1}{2^{(n+2)\gamma}}$$

$$|X_t - X_s| \le \frac{2}{2^{(n+1)\gamma}} + \frac{2}{2^{(n+2)\gamma}} + \frac{2}{2^{(n+3)\gamma}}$$

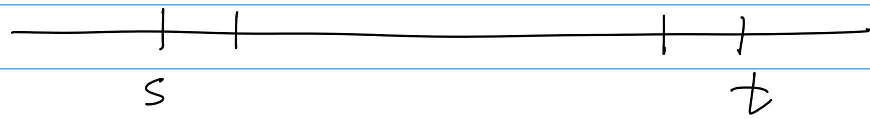
— — - — - — fine di consid.

$$\boxed{|X_t - X_s| \le 2 \sum_{k=n+1}^{m} \frac{1}{2^{k\gamma}}}$$

<u>induzione</u>

$s,t \in D_m$ $\qquad s \le s^* \qquad t^* \le t$

$s^* \in D_{m-1} \qquad t^* \qquad t^* \in D_{m-1}$



$$|X_t - X_s| \le |X_t - X_{t^*}| + |X_{t^*} - X_{s^*}| + |X_{s^*} - X_s|$$

$$\le \frac{2}{2^{m\gamma}}$$