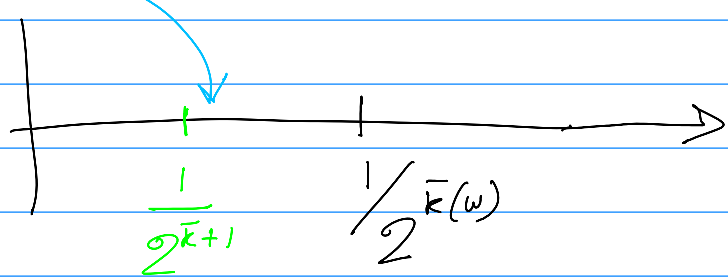
$\delta < \left(\frac{1}{2^n}\right)$ $\qquad \omega \notin A \qquad \bar{k}(\omega)$

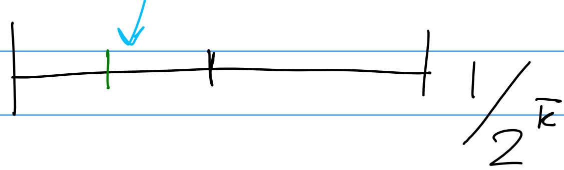
$|t-s| < \delta \qquad \delta \le \frac{1}{2^{\bar{k}(\omega)}}$

$$|t-s| < \delta \le \frac{1}{2^{\bar{k}(\omega)}}$$

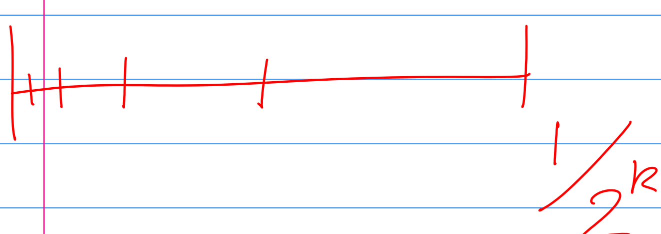


T

$$\frac{1}{2^{\bar{k}+1}} \le |t-s| < \frac{1}{2^{\bar{k}}}$$



$\exists\, n$ tale che $\quad \frac{1}{2^{n+1}} \le |t-s| < \frac{1}{2^n}$



$$|t-s| < \delta \qquad \frac{1}{2^{n+1}} \leq |t-s| < \frac{1}{2^n}$$

$t, s \in D$

$$|X_t - X_s| \leq 2 \sum_{k=n+1}^{m} \frac{1}{2^{k\gamma}}$$

$$\leq 2 \sum_{k=n+1}^{\infty} \frac{1}{2^{k\gamma}}$$

$$= 2 \, \frac{1}{2^{(n+1)\gamma}} \, \frac{1}{1 - 1/2^{\gamma}}$$

$$= \frac{2^{\gamma+1}}{2^{\gamma} - 1} \cdot \frac{1}{2^{(n+1)\gamma}}$$

$$|t-s| < \delta = \frac{1}{2^{k}} \qquad C = \frac{2^{\gamma+1}}{2^{\gamma}-1}$$

$$|X_t - X_s| \leq C \frac{1}{2^{(n+1)\gamma}} \qquad \exists n : \frac{1}{2^{(n+1)}} \leq |t-s| < \frac{1}{2^n}$$

$$\left[\frac{1}{2^{(n+1)}}\right]^{\gamma} \leq |t-s|^{\gamma}$$

$\delta = \frac{1}{2^{\bar{k}}}$ $|t-s| < \delta$ e diadici

$\Rightarrow$ $|X_t - X_s| \leq C\,|t-s|^\gamma$ $\underline{\underline{\omega \notin A}}$

$\omega \notin A$

$$X'_t(\omega) = \begin{cases} X_t(\omega) & \text{se } t \in D \\ \lim_{t_n \to t} X'_{t_n}(\omega) & \text{se } t \notin D \end{cases}$$

$t \notin D$ $t_n \longrightarrow t$ $t_n \in D$

$|t_{n+p} - t_n| \longrightarrow 0$ $n \longrightarrow \infty$ $\forall p$

$$|X'_{t_{n+p}} - X'_{t_n}| = |X_{t_{n+p}} - X_{t_n}| \leq C\,|t_{n+p} - t_n|^\gamma$$

$\{X'_{t_n}\}$ è di Cauchy $X'_{t_n} \longrightarrow$ numero $=: X'_t$

$\omega \in A$ $X'_t(\omega) = 0$

$\forall t \in [0,T]$ $X'_t(\omega)$ $\forall \omega$

1. X'_t è una $\underrightarrow{\text{v.c.}}$ $\quad X_t \quad t \in D$

$X_{t_n} \uparrow \quad t \notin D$

2. $X_t \overset{eq.}{=} X'_t \qquad t \in D$

$\mathbb{P}(X_t \neq X'_t) = 0 \quad ?$

$$(X_t \neq X'_t) = (X_t \neq X'_t) \cap \Omega$$

$$= \underbrace{\left[(X_t \neq X'_t) \cap A\right]}_{\subseteq A} \cup \underbrace{\left[(X_t \neq X'_t) \cap A^c\right]}_{= \emptyset}$$

2b. $t \notin D \qquad t_n \to t$

(I) $X_{t_n} \longrightarrow X'_t \qquad$ q.o.

(II) $\mathbb{E}(|X_t - X_s|^\alpha) \leq C\,|t-s|^{1+\beta}$

$$\mathbb{P}(|X_t - X_s| > \varepsilon) \leq \frac{1}{\varepsilon^\alpha}|t-s|^{1+\beta}$$

$$\mathbb{P}\left(|X_{t_n} - X_t| > \varepsilon \right) \leq \frac{1}{\varepsilon^{\alpha}} \underbrace{|t_n - t|^{1+\beta}}_{\downarrow \, 0}$$

$X_{t_n} \longrightarrow X_t$ <u>in probabilità</u>

q.c $\Longrightarrow$ convergenza in probabilità

(I) $X_{t_n} \longrightarrow X'_t$ in prob.

(II) $X_{t_n} \longrightarrow X_t$ in prob.

$m(t) = 0$

$\varphi(t,s) = t \wedge s$ $\Big| \Longrightarrow$ $W^0_t : \Big(\underbrace{\mathbb{R}^{[0,T]}}_{\Omega}, \underbrace{\sigma(\mathcal{C})}_{\mathcal{E}}, \mathcal{W}^0 \Big)$

$$W^0_t(\omega) := \omega(t)$$

$$A_{t_1 t_2 \dots t_n} = \begin{pmatrix} t_1 & t_1 & \dots & t_1 \\ \vdots & t_2 & \dots & t_2 \\ \vdots & | & & \\ t_1 & t_2 & \dots & t_n \end{pmatrix}$$

$$\mu_{t_1 t_2 \dots t_n}(B) = \int_B \left(\frac{1}{\sqrt{2\pi}}\right)^n \frac{1}{\sqrt{\det A}} e^{-\frac{1}{2}\langle Ax, x\rangle} dx \qquad \mathbb{R}^n$$

$$\{ q \in \mathbb{R}^{[0,T]} : (q(t_1) \dots q(t_n)) \in B \} = C_{t_1 t_2 \dots t_n, B} \subset \Omega$$

$$\mathcal{W}^\circ\left(C_{t_1 t_2 \dots t_n, B}\right) := \mu_{t_1 t_2 \dots t_n}(B)$$

$$\mathbb{E}_{\mathcal{W}^\circ}\left(\left| W^\circ_t - W^\circ_s \right|^{2m}\right) = C_m |t-s|^m$$

$$\|$$

$$\int_{\mathbb{R}^{[0,T]}} \left| \omega(t) - \omega(s) \right|^{2m} \mathcal{W}^\circ(d\omega)$$

$$\|$$

$$\int_{\mathbb{R}^2} |x_1 - x_2|^{2m} \left(\frac{1}{\sqrt{2\pi}}\right)^2 \frac{1}{\sqrt{\det A}} e^{-\frac{1}{2}\langle A^{-1}x, x\rangle} dx$$

$$x = (x_1, x_2)$$

$$= \int_{\mathbb{R}^2} |x_1 - x_2|^{2m} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x_1^2}{2s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x_2-x_1)^2}{2(t-s)}} dx_1 dx_2$$

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il teorema di regolarità ci assicura l'esistenza di un processo equivalente a W°_t

W_t

dove W_t è a traiettorie continue (hölderiane di $\gamma < \frac{1}{2}$)

$W_t^\circ(\omega) = \omega(t)$ $\qquad \left(\mathbb{R}^{[0,T]}, \sigma(\mathcal{C}), \underline{\underline{\mathcal{W}^\circ}}\right)$

$W_t(\omega)$ è a traiettorie continue

$\{X_t\}$ $(\Omega, \mathcal{E}, \mathbb{P})$

$X: \Omega \longrightarrow \mathbb{R}^{[0,T]}$

$X_t : \Omega, \mathcal{E}, \mathbb{P}$ $\quad$ <u>è a traiettorie continue</u>

$X: \Omega \longrightarrow C([0,T], \mathbb{R})$

$\mathcal{C}$ $\qquad C_{t_1 t_2 \ldots t_n, B} = \left\{ q \in C([0,T], \mathbb{R}) : (q(t_1) \ldots q(t_n)) \in B \right\}$

$\sigma(\mathcal{C})$

Q_X misura immagine di $\mathbb{P}$ attraverso X

$$\left(C([0,T], \mathbb{R}), \underset{\mathcal{B}}{\underset{|||}{\sigma(\mathcal{C})}}\right)$$

$X : \Omega \longrightarrow C([0,T], \mathbb{R})$

$X' : \Omega' \longrightarrow C([0,T], \mathbb{R})$

$$\mathcal{M}_6 = \mathcal{M}_6' \Longrightarrow Q_X \equiv Q_{X'}$$

$W_t^0 \quad (\mathbb{R}^{[0,T]}, \sigma(\mathcal{C}), \mathcal{W}^0)$

W_t a traiett. continue

La misura immagine di $\mathcal{W}^0$ tramite (W_t) su $C([0,T], \mathbb{R})$

$\mathcal{W}$ misura di Wiener (1923)