Probabilità condizionata

$(\Omega, \mathcal{E}, \mathbb{P})$ $\quad\quad$ $\mathbb{P}$ originale

$$\boxed{(\Omega, \mathcal{E}, \mathbb{P}')}$$

E

$\mathbb{P}(E)$ $\times$

$\nexists\ E$

$\mathbb{P}(E|F)$ $\times$

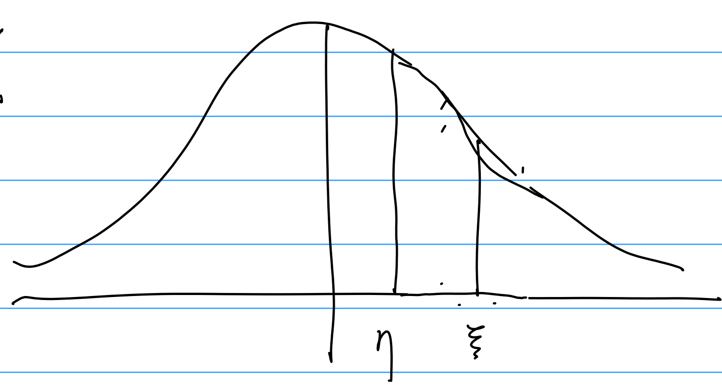
HP $\quad \underbrace{\mathbb{P}(E \cap F)}_{\text{originale}} = \underbrace{\mathbb{P}(E|F)}_{\text{nuova}} \cdot \underbrace{\mathbb{P}(F)}_{\text{originale}}$

$$\mathbb{P}(E|F) := \frac{P(E \cap F)}{P(F)} \qquad P(F) > 0$$

ω

$\boxed{X(\omega)} = \xi$

X



$\{\eta, \xi\}$

$\mathbb{P}(F) = 0 \qquad \mathbb{P}(E|F)$

$P(E|F) \qquad P(E|F')$

Partiamo dalla fine (quasi) $(\Omega, \mathcal{E}, P)$

E, σ-algebra $\mathcal{F} \subset \mathcal{E}$

$\{\emptyset, \Omega\} \dots \mathcal{F} \dots \mathcal{E}$

$\mathbb{P}(E|\mathcal{F}) : \Omega \longrightarrow [0,1]$

variabile casuale tale che

$Z = \mathbb{P}(E|\mathcal{F}) \qquad Z^{-1}(I) \in \mathcal{E}$

1. è $\mathcal{F}$-misurabile $\left(Z^{-1}(I) \in \mathcal{F} \right)$

2. $$P(E \cap F) = \int_F \mathbb{P}(E|\mathcal{F}) \, \mathbb{P}(d\omega)$$

per ogni $F \in \mathcal{F}$.

Unicità di $\mathbb{P}(E|\mathcal{F})$ quasi ovunque

Dim. $\exists$ Z Z' $\mathcal{F}$-misurabili tali che

$$P(E\cap F) = \int_F Z\,dP = \int_F Z'\,dP$$

$$\int_F (Z - Z')\,dP = 0$$

$Z - Z'$ è ancora $\mathcal{F}$-misurabile

W $\mathcal{F}$ misurabile tale che

$$\int_F W\,dP = 0 \quad \forall F \in \mathcal{F}$$

$\Rightarrow$ $W = 0$ q.o.

1933

$\mathcal{F}_0 = \{\emptyset, \Omega\}$

$F = \emptyset$

$F = \Omega$

$E \in \mathcal{E}$

$$P(E \mid \mathcal{F}_0) = c$$

$$\cancel{0 = P(E \cap \emptyset) = \int_{\emptyset} c \, dP = 0}$$

$$P(E) = P(E \cap \Omega) = \int_{\Omega} c \, dP = c$$

$$P(E \mid \mathcal{F}_0) = P(E)$$

$$\mathcal{F}_1 = \{ \emptyset, F, F^c, \Omega \} \qquad \begin{array}{l} P(F) > 0 \\ P(F^c) > 0 \end{array}$$

$E \in \mathcal{E}$

1. $\mathcal{F}_1$ - <u>misurabili</u> sono le costanti

$$Z(\omega) = \begin{cases} c_1 & \text{su } F \\ c_2 & \text{su } F^c \end{cases}$$

$\cancel{\emptyset}$

F $\qquad P(E \cap F) = \int_F Z \, dP = c_1 P(F)$

F^c $\qquad P(E \cap F^c) = \int_{F^c} Z \, dP = c_2 P(F^c)$

Ω automaticamente verificato

$$c_1 = \frac{P(E\cap F)}{P(F)} = P(E|F)$$

$$c_2 = \frac{P(E\cap F^c)}{P(F^c)} = P(E|F^c)$$

$$P(E\cap\Omega) = \int_\Omega Z\,dP = \int_F P(E|F)\,dP + \int_{F^c} P(E|F^c)\,dP$$

$$= \underline{P(E|F)\,P(F)} + \underline{P(E|F^c)\,P(F^c)}$$

$$\mathcal{F}_1 = \{\emptyset, F_1, F_2, \Omega\}$$

$F_1 \cup F_2 = \Omega$

$F_1 \cap F_2 = \emptyset$

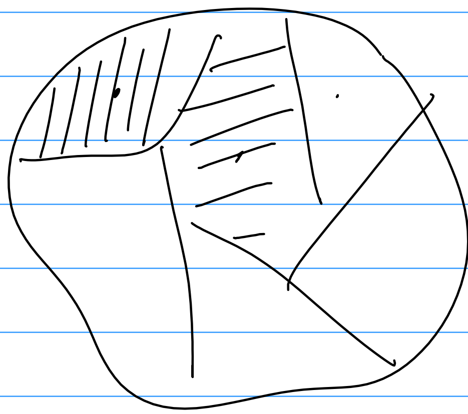
$$\mathcal{F}_2 = \{\emptyset, F_1, F_2, F_3, F_3^c, F_1^c, F_2^c, \Omega$$

$$\mathcal{F} = \{\emptyset, F_1, F_2, \dots, F_n, \dots\dots, \Omega\}$$

$F_i \cap F_j = \emptyset$, $i \neq j$

$\bigcup F_i = \Omega$

$\omega \in F_k$ $\qquad P(E|\mathcal{F})(\omega) = P(E|F_k)$



$P(E|\mathcal{F})(\omega)$

16/3/09

$$\mathbb{E}(X|\mathcal{F})$$

0. $X \in L^1(\Omega)$

$$\mathbb{E}(X|\mathcal{F})$$

è l'unica v.c.

1. $\mathcal{F}$-misurabile (e in L^1)

2. $$\int_F X(\omega)\, \mathbb{P}(d\omega) = \int_F \mathbb{E}(X|\mathcal{F})(\omega)\, \mathbb{P}(d\omega)$$

$\forall F \in \mathcal{F}$.

Unie.

$$\int_F X(\omega)\,\mathbb{P}(d\omega) = \underbrace{\int_F Z\,dP = \int_F Z'\,dP}$$

$$\int_E \left(X - \mathbb{E}(X|\mathcal{F})\right) dP = 0 \qquad \forall E \in \mathcal{E}$$

$$\boxed{\begin{array}{ll} \mathbb{P}(A|\mathcal{F}) & \\ \uparrow & \\ \mathbb{E}(X|\mathcal{F}) & X \in L^1 \end{array}}$$

$A \in \mathcal{E} \qquad X = \mathbb{1}_A$

$$\mathbb{E}(\mathbb{1}_A|\mathcal{F})$$

$$\int_F \mathbb{1}_A\,dP = \int_F \mathbb{E}(\mathbb{1}_A|\mathcal{F})\,dP$$

$$\|$$

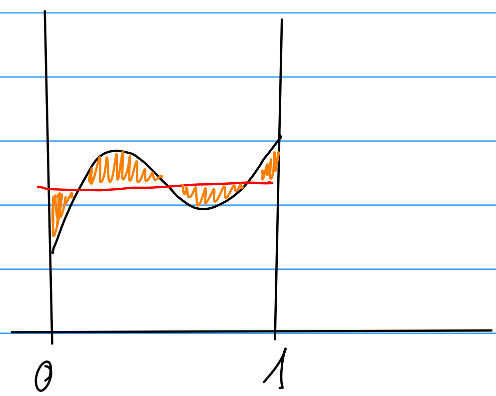
$$P(A \cap F) = \int_F \mathbb{P}(A|\mathcal{F})\,dP$$

Esempi:

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \qquad X \in L^1$$

$$\mathbb{E}(X \mid \mathcal{F}) = \mathbb{E}(X)$$

$$\mathbb{E}(X) = \int_\Omega X \, dP = \int_\Omega \mathbb{E}(X \mid \mathcal{F}) \, dP = c$$



$\Omega = (0,1)$

$\mathcal{B}$

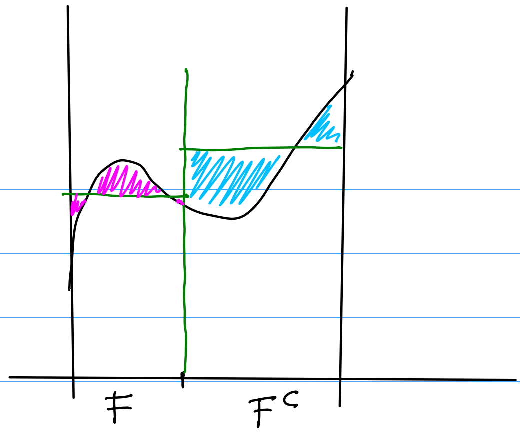
dx

2. $$\mathcal{F}_1 = \{\emptyset, F, F^c, \Omega\}$$

$$\mathbb{E}(X \mid \mathcal{F})(\omega)$$

$$\frac{1}{P(F)} \int_F X \, dP = c \quad \text{su } F$$

$$\frac{1}{P(F^c)} \int_{F^c} X \, dP = c' \quad \text{su } F^c$$



Proprietà 1. Se X è $\mathcal{F}$-misurabile (L^1)

$$\mathbb{E}(X|\mathcal{F}) = X \quad \text{q.o.}$$

Proprietà 2. $X \in L^1(\Omega)$

e indipendente da $\mathcal{F}$

— $X^{-1}(I)$, $I \in \mathcal{B}$ $\quad \mathcal{F}_X = \{X^{-1}(I), I \in \mathcal{B}\}$

σ-algebra $\subseteq \mathcal{E}$

$\mathcal{F}_X$ è la più piccola σ-algebra che rende X misurabile.

$\mathcal{F}_X$ e $\mathcal{F}$ sono indipendenti

$$\begin{matrix} \Downarrow & \Downarrow \\ \forall A & \forall B \end{matrix} \qquad P(A \cap B) = P(A)P(B)$$

Prop. 2 $\Longrightarrow$ $\mathbb{E}(X|\mathcal{F}) = \mathbb{E}(X)$

Dimostrazione.

<u>1º passo</u> $X = \mathbb{1}_A$ $\mathcal{F}_X = \{\emptyset, A, A^c, \Omega\}$

$\forall F \in \mathcal{F}$ $P(A \cap F) = P(A)P(F)$

$$\int_F X \, dP = \int_F \mathbb{1}_A \, dP = P(A \cap F) = P(A)P(F)$$

$$\parallel$$

$$\int_F \mathbb{E}(X|\mathcal{F}) \, dP = \int_F P(A) \, dP$$

$$\mathbb{E}(X|\mathcal{F}) = P(A) = \mathbb{E}(X)$$

2º passo.

$$X = \sum_{i=1}^{m} \alpha_i \mathbb{1}_{A_i}$$

$\mathcal{F}_X = \{\emptyset, A_1, A_2 \dots A_n, \dots, \Omega\}$ $P(A_i \cap F) = P(A_i)P(F)$

$P((A_1 \cup A_2) \cap F) = P(A_1 \cup A_2)P(F)$

$$\int_F X \, dP = \sum_{i=1}^{n} \alpha_i P(A_i \cap F) = \underbrace{\left[\sum_{i=1}^{n} \alpha_i P(A_i)\right]}_{\mathbb{E}(X)} P(F)$$

$$= \int_F \mathbb{E}(X) \, dP$$

<u>3° passo</u> $X \geq 0$ $\mathcal{E}$-misurabile

$$X_1 \leq X_2 \leq \ldots \leq X_n \leq \ldots \uparrow X$$

$\mathcal{F}_X$ - misurabile $\subset \mathcal{E}$

$$\mathbb{E}(X_i | \mathcal{F}) = \mathbb{E}(X_i)$$

$$\mathbb{E}(X_1) \leq \mathbb{E}(X_2) \leq \mathbb{E}(X_3) \leq \ldots \leq \mathbb{E}(X)$$

Proprietà 3. $X \leq Y$ $X, Y \in L^1$

$$\Longrightarrow \mathbb{E}(X|\mathcal{F}) \leq \mathbb{E}(Y|\mathcal{F})$$

$$X_1 \leq X_2 \leq \ldots \leq X_n \qquad \boxed{X_n \uparrow X}$$

$$\forall \quad \mathbb{E}(X_1|\mathcal{F}) \leq \mathbb{E}(X_2|\mathcal{F}) \ldots \leq \mathbb{E}(X|\mathcal{F})$$

Prop.

$$\mathbb{E}(X_n | \mathcal{F}) \longrightarrow \mathbb{E}(X|\mathcal{F})$$

Spesso $\quad X \in L^1 \qquad X = X^+ - X^-$

Prop. 5. $\quad \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{E} \qquad X \in L^1$

$$\mathbb{E}\big(\mathbb{E}(X|\mathcal{F}_2)\,\big|\,\mathcal{F}_1\big) = \mathbb{E}(X|\mathcal{F}_1)$$

Dimostr.

$$\int_{F_1} \mathbb{E}(X|\mathcal{F}_2)\,dP = \int_{F_1} \mathbb{E}\big(\mathbb{E}(X|\mathcal{F}_2)\,\big|\,\mathcal{F}_1\big)\,dP$$

$$\|$$

$$\int_{F_1} X\,dP = \int_{F_1} \mathbb{E}(X|\mathcal{F}_1)\,dP$$

Esistenza: Teorema di Radon-Nykodim

$$\mu \ll \nu \qquad \nu(E) = 0 \Rightarrow \mu(E) = 0$$

$$\mu(E) = \int_E \varphi(x)\,\nu(dx)$$

$\uparrow$ esiste questa φ

$X \in L^1$, $\mathcal{F} \subset \mathcal{E}$ $\quad \mathbb{E}(X|\mathcal{F})$ c'è o no?

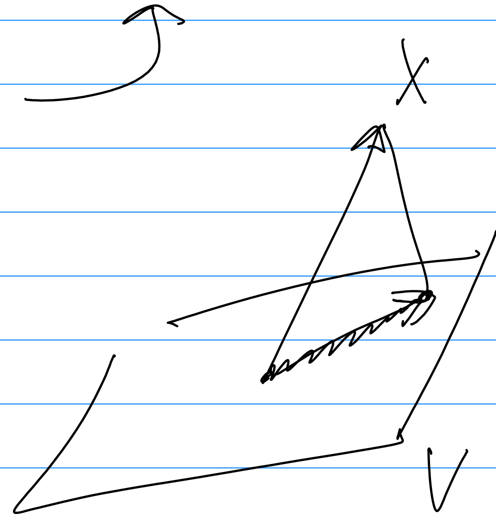
1° passo: $X \in L^2(\Omega) \subset L^1(\Omega)$

$L^2(\Omega)$ è uno spazio di Hilbert $\quad L^2(\Omega, \mathcal{E}, \mathbb{P}; \mathbb{R})$

$$L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$$

$$V = L^2(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{R}) \subset L^2(\Omega, \mathcal{E}, \mathbb{P}, \mathbb{R})$$

sottospazio di Hilbert



$X \in L^2(\Omega, \mathcal{E}, P)$

$\mathbb{E}(X|\mathcal{F})$