H 1) sono spazi vettoriali $\begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$

prodotto scalare

(x, y) forma bilineare nel caso reale

$$(\lambda x + x', y) = \lambda (x, y) + (x', y) \qquad (x, y) \in \mathbb{R}$$
$$(x, \lambda y + y') = \lambda (x, y) + (x, y')$$

nel caso complesso $\lambda \in \mathbb{C}$

$$(x + x', y) = (x, y) + (x', y) \qquad (x, y) \in \mathbb{C}$$
$$(x, y + y') = (x, y) + (x, y')$$

sesquilineare

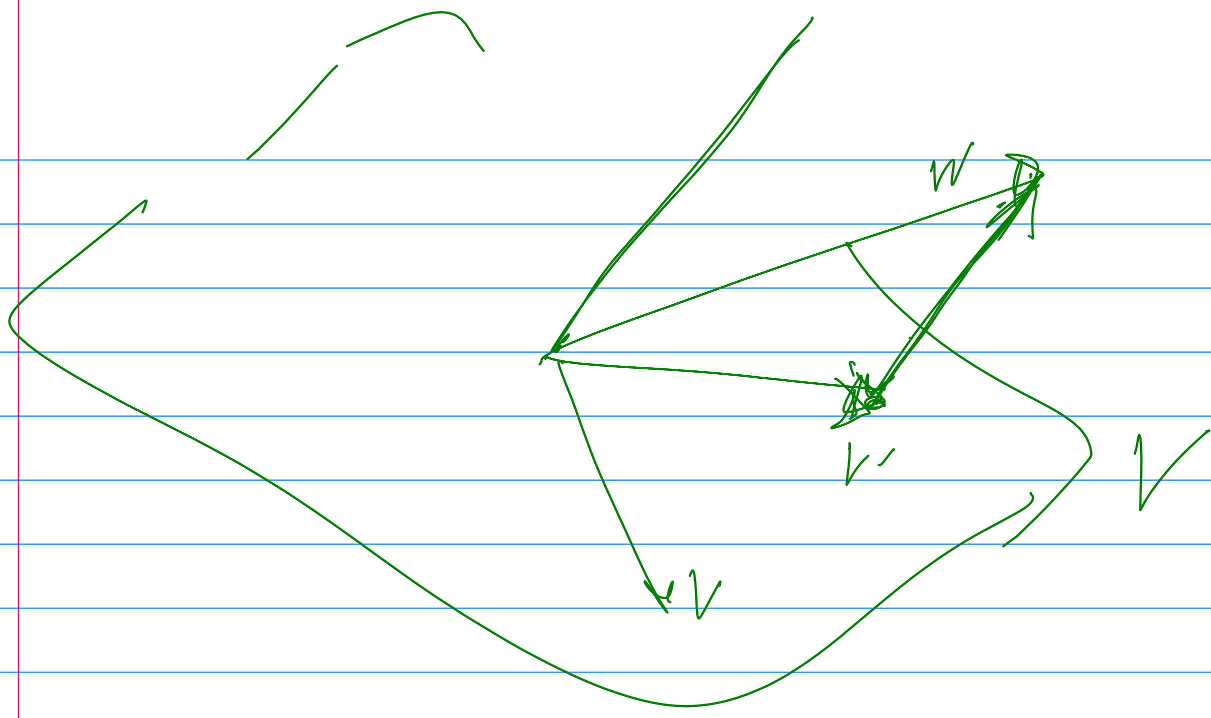
$$(\lambda x, y) = \lambda (x, y)$$
$$(x, \lambda y) = \bar{\lambda} (x, y)$$

- $(x, y) = \overline{(y, x)}$ - $(x, x) \geq 0$

- $(x, x) = 0 \implies x = 0$

Prop. $\|x\| = \sqrt{(x, x)}$

- H è completo -



$$(w - v', v) = 0$$

$$(w, v') = (w, v) \quad \forall v \in V$$

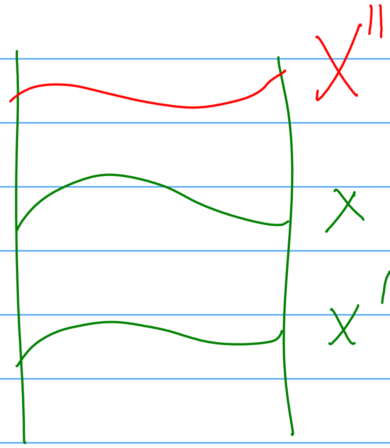
$$H = L^2(\Omega, \mathcal{E}, \mathbb{P}) = \{X : \int_\Omega |X|^2 dP < +\infty\}$$

1. $\mathcal{E}$-misurabili

$$(X, Y) = \int_\Omega X(\omega) Y(\omega) P(d\omega)$$

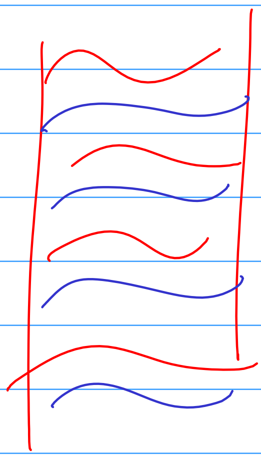
$$V = L^2(\Omega, \mathcal{F}, P) = \{ \ldots$$

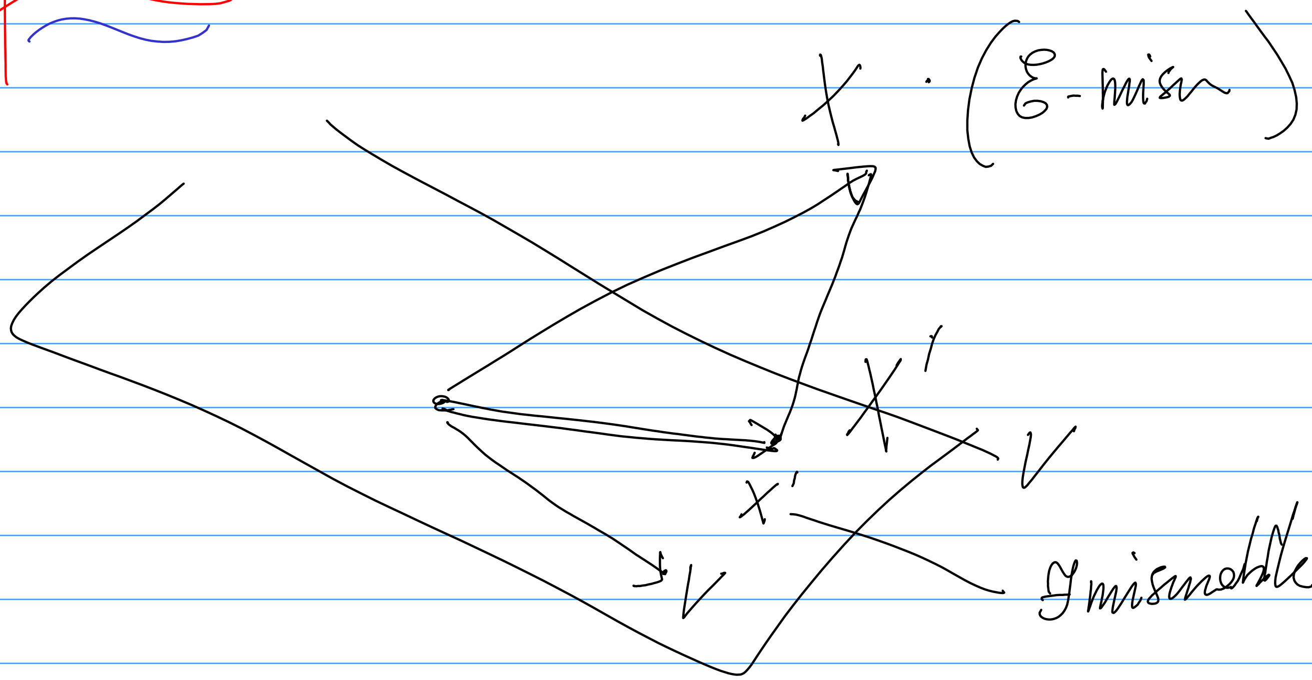
$\mathcal{F}$-misurabili



$$V \subset \tilde{H}$$

$0 \in H \qquad 0 \in V$





$$(X', V) = (X, V)$$

$$\int_\Omega X'(\omega)\,V(\omega)\,dP = \int_\Omega X(\omega)\,V(\omega)\,dP$$

$$V = \mathbb{1}_F \qquad F \in \mathcal{F} \subset \mathcal{E}$$

$$\int_F X\,dP = \int_F X'\,dP$$

$$\int_\Omega \left| \mathbb{E}(X \mid \underline{\underline{\mathcal{F}}}) \right|^2 dP \leq \int_\Omega |X|^2\,dP$$

<u>sospendo</u>

$$L^2(\Omega, \mathcal{E}, P) \subset L^1(\Omega, \mathcal{E}, P)$$

$$X \in L^1 \qquad X_n \in L^2$$

$$\int_\Omega |X_n - X|\,dP \longrightarrow 0$$

$X \geq 0 \qquad L^1 \qquad X_n \uparrow X$

$X_n \in L^\infty \subset L^2$

$X_n \leq X_{n+1}$

$X'_n \leq X'_{n+1}$

esercizio

$\mathbb{E}(X_n | \mathcal{F}) \leq \mathbb{E}(X_{n+1} | \mathcal{F}) \uparrow \underline{\underline{\mathbb{E}(X|\mathcal{F})}}$

$X = X^+ - X^-$

$\mathcal{F} = \mathcal{F}_Y = \{ Y^{-1}(I) , I \in \mathcal{B} \}$

$X \in L^1 \qquad \mathbb{E}(X | \mathcal{F}_Y)$

$= \mathbb{E}(X|Y)$

1. $\mathcal{F}_Y$ - misurabile

2. $\displaystyle\int_{Y^{-1}(I)} X \, dP = \int_{Y^{-1}(I)} \mathbb{E}(X|Y) \, dP$ $\qquad$ Y - misurabile

$\mathbb{E}(X|Y) = \varphi(Y)$

Esercizio: Z è Y - misurabile

$\varphi : \mathbb{R} \to \mathbb{R}$ di Borel

$Z = \varphi(Y)$ $\qquad \varphi(Y)^{-1}(I) = Y^{-1}(\varphi^{-1}(I))$

se e solo esiste $\varphi : \mathbb{R} \to \mathbb{R}$ boreliana tale che $Z = \varphi(Y)$

Dimostrazione.

1° passo $\quad Z = \mathbb{1}_A \quad : \quad Z$ è Y misurab.

$$\Rightarrow \quad A = Y^{-1}(I)$$

$$Z = \mathbb{1}_{Y^{-1}(I)}(\omega) = \mathbb{1}_I(Y(\omega))$$

2ª $\quad \sum \alpha_i \mathbb{1}_{A_i} \qquad \varphi = \sum \alpha_i \mathbb{1}_{I_i}$

3^a $\quad Z \geq 0 \quad Y$ - misurabile

$Z_n \uparrow Z \qquad Z_n$ semplice e Y-mis

$\varphi_n(Y) \uparrow Z \qquad = \varphi_n(Y)$

$\varphi = \limsup \varphi_n \qquad \varphi_n : \mathbb{R} \to \mathbb{R}$

$= \liminf \varphi_n \qquad \varphi_n(Y) \to \varphi(Y)$

$\varphi(Y) = Z$

$Z = Z^+ - Z^-$

<u>Esercizio</u> : $\quad Z \in \mathbb{R}^k \quad Y \in \mathbb{R}^n$

$Z = \varphi(Y) \quad \varphi : \mathbb{R}^n \to \mathbb{R}^k$

<u>Applicazione</u>

$$\int_{Y^{-1}(I)} X \, dP = \int_{Y^{-1}(I)} \varphi(Y) \, dP$$

$$\int_{Y^{-1}(I)} X \, dP = \int_{\Omega} \mathbb{1}_{Y^{-1}(I)} \varphi(Y) \, dP =$$

$$= \int_{\Omega} \mathbb{1}_I (Y) \varphi(Y) \, dP$$

$$\mathbb{1}_I (y) \varphi(y) \overset{!}{=}: F(y) \qquad \int_{\Omega} F(Y) \, dP =$$

$$= \int_{\mathbb{R}} F(y) \underbrace{\mathbb{P}(Y \in dy)}_{\mu_Y (dy)}$$

$$\int_{Y^{-1}(I)} X \, dP = \int_{I} \cdots \varphi(y) \, \mathbb{P}(Y \in dy)$$

$$\varphi(y) = \mathbb{E}(X \mid Y = y)$$

$$\int_{Y^{-1}(I)} X \, dP = \int_{I} \mathbb{E}(X \mid Y = y) \, \mathbb{P}(Y \in dy)$$

$X = \mathbb{1}_A \qquad \mathbb{E}(\mathbb{1}_A \mid Y = y)$

$= \mathbb{P}(A \mid Y = y)$

$$\mathbb{P}(A \cap Y^{-1}(I)) = \int_I \mathbb{P}(A \mid Y = y)\, \mathbb{P}(Y \in dy)$$

$$\mathbb{E}(X \mid Y) = \mathbb{E}(X \mid Y = y) \circ Y$$

$\{ X_t , t \in [0,T] \}$

$\mathcal{F}_t$ = la più piccola σ-algebra per la quale ogni X_s, $s \leq t$ è misurabile.

$X_s \qquad \mathcal{F}_{X_s} \subset \mathcal{F}_t \qquad$ se $\quad s \leq t$

$\mathcal{F}_t = \sigma(X_s , s \leq t) \qquad \bigcup_{s \leq t} \mathcal{F}_{X_s} \subsetneq \mathcal{F}_t$

$\{ \mathcal{F}_t , t \in [0,T] \}$

$\{\omega : (X_{t_1}, X_{t_2}, \dots, X_{t_n}) \in I\} \subset \Omega$

$t_1\, t_2 \dots t_n \leq t$

$(\Omega, \mathcal{E}, P)$ $(X_t, t \leq T)$ traiettorie siano continue

$X : \Omega \longrightarrow C([0,T], \mathbb{R})$

$$\boxed{\begin{array}{c}\sigma(\mathcal{C}) \\ \parallel \\ \mathcal{B}\end{array}}$$

Q_X è la misura immagine

$\left(C([0,T];\mathbb{R}), \mathcal{B}, Q_X\right)$

$\omega \leftrightarrow q$

$\{\tilde{X}_t, t \leq T\}$

$\tilde{X}_t(q) := q(t)$

processo canonico

$\tilde{\mathcal{F}}_t = \sigma\{\tilde{X}_s \quad s \leq t \quad (t \leq T)\}$

X_t $\qquad \mathcal{F}_t = \sigma(X_s, s \leq t)$

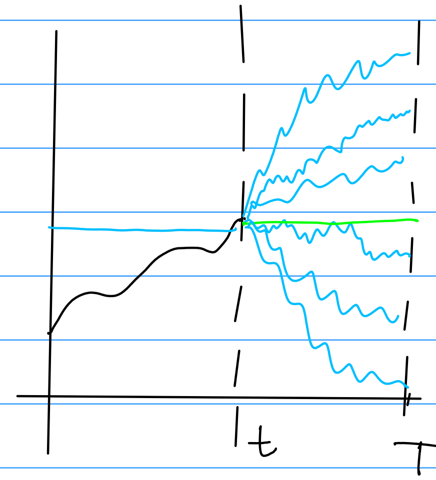
$\mathcal{F}_{X_s} \subset \mathcal{F}_t \qquad t_1, t_2 \leq t$

$\sigma(\mathcal{F}_{X_{t_1}}, \mathcal{F}_{X_{t_2}}) = \sigma\{X_{t_1}^{-1}(I) \cap X_{t_2}^{-1}(J)\}$

$\{\omega : X_{t_1} \in I, X_{t_2} \in J\}$

$$\mathcal{F}_t = \sigma\{(X_{t_1}, X_{t_2} \dots X_{t_n}) \quad t_k \le t\}$$

$$\tilde{\mathcal{F}}_t = \sigma(\mathcal{C}_t)$$



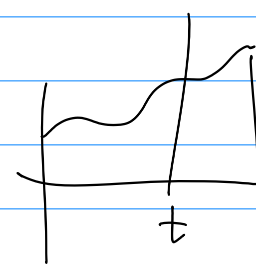
$C([0,t], \mathbb{R})$ $\quad \mathcal{B}_{[0,t]}$

$$C([0,T] : \varphi(s) = 0,\ s \le t) = C_t.$$

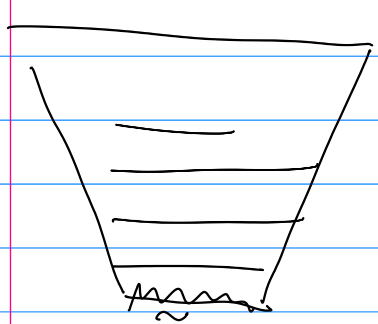
$\psi \in C([0,t], \mathbb{R})$ $\quad \tilde{\psi} =$ prolungata costante

$\tilde{\psi} + C_t$

$$\tilde{\mathcal{F}}_t = \{\tilde{\psi} + C_t,\ \psi \in \mathcal{B}_{[0,t]}\}$$



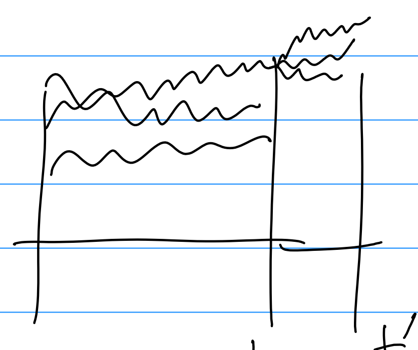
$$\tilde{\mathcal{F}}_0 = \{I + C_0,\ I \in \mathcal{B}_{\mathbb{R}}\}$$



T

$\tilde{\mathcal{F}}_0$

$$\mathcal{F}_T \equiv \mathcal{B}_{[0,T]}$$



cambio notazione

$\boxed{\mathring{\mathcal{F}}_t}$

$\downarrow$

X_t

$\mathring{\mathcal{F}}_t \subset \mathcal{F}_t$

$\downarrow$

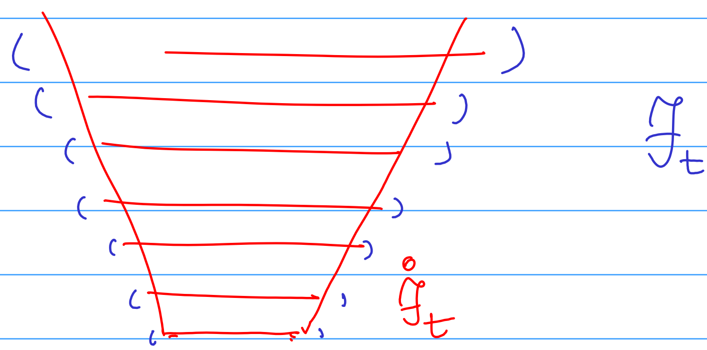
il processo è adattato alla filtrazione

per ogni t $\quad X_t$ è $\mathcal{F}_t$-misurabile

$\boxed{\begin{array}{c}\mathcal{F}_t \subset \mathcal{F}_{t'} \\ \underline{\text{filtrazione}}\end{array}}$

$\{X_t\}$ è adattato a $\mathring{\mathcal{F}}_t$? sì

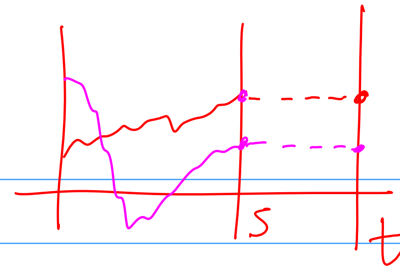
filtrazione naturale



Definizione: Un processo X_t è una martingala rispetto ad una filtrazione $(\mathcal{F}_t)$ se

0. $X_t \in L^1(\Omega) \qquad \forall t \leq T$
1. $\{X_t\}$ è adattato alla filtrazione $\{\mathcal{F}_t\}$
2. $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s \quad (9.0) \qquad s \leq t$

Condizionare a $\mathcal{F}_s$



Esempio $\quad Z \quad \mathcal{E}$- misurabile $\quad Z \in L^1$

$(\mathcal{F}_t)$ filtrazione

Def. $\quad X_t := \mathbb{E}(Z \mid \mathcal{F}_t)$

X_t è una martingala rispetto a $(\mathcal{F}_t)$?

0.) $\mathbb{E}(Z \mid \mathcal{F}) \in L^1 \qquad \mathbb{E}\big(\mathbb{E}(Z \mid \mathcal{F})\big) = \mathbb{E}(Z)$

1.) è adattato a $(\mathcal{F}_t)$

2. $\mathbb{E}(X_t \mid \mathcal{F}_s) = \mathbb{E}\big(\mathbb{E}(Z \mid \mathcal{F}_t) \mid \mathcal{F}_s\big) = \mathbb{E}(Z \mid \mathcal{F}_s) =: X_s$

Supponete che (X_t) $(\mathcal{F}_t)$ 0. 1. e

$$2' : \quad \mathbb{E}(X_t \mid \mathcal{F}_s) \geq X_s \qquad \text{q.o. } s \leq t$$

si dice sottomartingala

$\left\{ \text{sopramartingala} \qquad \geq s \right\}$

Esercizi:

W_t è una martingala $(\mathcal{F}_t)$

$$\mathbb{E}(W_t \mid \mathcal{F}_s) = W_s \qquad s \leq t$$

$$\mathbb{E}(W_t - W_s \mid \mathcal{F}_s) \overset{\circ}{=} \mathbb{E}(W_t - W_s) = 0$$

N_t $\quad \mathbb{E}(N_t \mid \mathcal{F}_s) = ?$

$$\mathbb{E}(N_t - N_s \mid \mathcal{F}_s) \overset{\circ}{=} \mathbb{E}(N_t - N_s) = \lambda(t-s)$$

$$\mathbb{E}(N_t \mid \mathcal{F}_s) = N_s + \lambda(t-s)$$

$$\mathbb{E}(N_t - \lambda t \mid \mathcal{F}_s) = N_s - \lambda s$$

W_t^2 è una martingala rispetto a $\mathcal{F}_t$

$$\mathbb{E}(W_t^2 \mid \mathcal{F}_s)$$

$$(W_t - W_s)^2 = W_t^2 - 2W_tW_s + W_s^2$$

$$\mathbb{E}\left((W_t - W_s)^2 \mid \mathcal{F}_s\right) = \mathbb{E}(W_t^2 \mid \mathcal{F}_s) - 2\,\mathbb{E}(W_sW_t \mid \mathcal{F}_s) + W_s^2$$

$\downarrow$

$$\mathbb{E}\left(\left(W_t - W_s\right)^2\right) = t - s$$

$$\mathbb{E}\left(W_t^2 \mid \mathcal{F}_s\right) = t - s + 2\,\mathbb{E}\left(W_s W_t \mid \mathcal{F}_s\right) - W_s^2$$

$$\mathbb{E}\left(XY \mid \mathcal{F}\right) = Y\,\mathbb{E}\left(X \mid \mathcal{F}\right) \qquad XY \in L^1$$

$$X \in L^1$$

Y è $\mathcal{F}$ – misurabile

$$= t - s + 2 W_s\,\mathbb{E}\left(W_t \mid \mathcal{F}_s\right) - W_s^2$$

$$= t - s + 2W_s^2 - W_s^2 = t - s + W_s^2$$

$$\mathbb{E}\left(W_t^2 - t \mid \mathcal{F}_s\right) = W_s^2 - s$$

$$\mathbb{E}\left[\left(N_t - \lambda t\right)^2 - \lambda t \mid \mathcal{F}_s\right) =$$

$$= \mathbb{E}\left(\left(N_t - \lambda t\right)^2 \mid \mathcal{F}_s\right) - \lambda t =$$

$$= \mathbb{E}\left(\left(N_t - N_s + N_s - \lambda t\right)^2 \mid \mathcal{F}_s\right) - \lambda t$$

$$= \mathbb{E}\left(\left(N_t - N_s\right)^2 \mid \mathcal{F}_s\right) + 2\,\mathbb{E}\left(\left(N_t - N_s\right)\left(N_s - \lambda t\right) \mid \mathcal{F}_s\right) +$$

$$+ \left(N_s - \lambda t\right)^2 - \lambda t$$

$$= \lambda(\cancel{t}-s) + \lambda^2(\cancel{t}-s)^2 + 2(N_s - \lambda t)\,\lambda(t-s) + (N_s-\lambda t)^2 - \cancel{\lambda t}$$

$$= -\lambda s + \left[N_s - \cancel{\lambda t} + \lambda(\cancel{t}-s)\right]^2$$

$$= (N_s - \lambda s)^2 - \lambda s$$

$$P(X_t \in I \mid \mathcal{F}_s) = P(X_t \in I \mid X_s)$$