$$\mathbb{E}((W_t^2 - t)^2 \mid \mathcal{F}_s)$$

$$Y := W_t - W_S$$

$$\mathbb{E}[(W_t^2 - t)^2 \mid \mathcal{F}_s] = \mathbb{E}[(W_s + Y)^2 - t)^2 \mid \mathcal{F}_s]$$

$$\mathbb{E}[(W_s^2 + 2W_sY + Y^2 - t)^2 \mid \mathcal{F}_s] = \mathbb{E}[((W_s^2 - s) + 2W_sY + Y^2 - t + s)^2 \mid \mathcal{F}_s]$$

$$\begin{aligned}((W_s^2 - s) + 2W_sY + Y^2 - t + s)^2 = (W_s^2 - s)^2 + 4W_s^2Y^2 + Y^4 + (t-s)^2 + \\ +4(W_s^2 - s)W_sY + 2(W_s^2 - s)Y^2 - 2(W_s^2 - s)(t-s) + \\ +4W_sY^3 - 4W_sY(t-s) - 2Y^2(t-s)\end{aligned}$$

$$\mathbb{E}((W_t^2 - t)^2 \mid \mathcal{F}_s) = (W_s^2 - s)^2 + 4W_s^2(t-s) + 2(t-s)^2$$

$$\mathbb{E}\Big(\int_0^t W_r^2\, dr \mid \mathcal{F}_s\Big) = \int_0^s W_r^2\, dr + (W_s^2 - s)(t-s) + \frac{t^2 - s^2}{2}$$

$$\mathbb{E}\Big(4\int_0^t W_r^2\, dr \mid \mathcal{F}_s\Big) - 4\int_0^s W_r^2\, dr = 4W_s^2(t-s) + 2(t-s)^2$$

$$\mathbb{E}\Big((W_t^2 - t)^2 - 4\int_0^t W_r^2\, dr \mid \mathcal{F}_s\Big) = (W_s^2 - s)^2 - 4\int_0^s W_r^2\, dr$$

Osservazione (di Francesco Romeo — Romeo è il cognome). Se consideriamo il processo

$$B_t = 4tW_t^2 - 2t^2$$

e calcoliamo

$$\mathbb{E}(B_t \mid \mathcal{F}_s) = 4t(W_s^2 + t - s) - 2t^2 = 4tW_s^2 + 2t^2 - 4ts = 4tW_s^2 + 2(t-s)^2 - 2s^2$$

$$\mathbb{E}(B_t \mid \mathcal{F}_s) = 4(t-s)W_s^2 + 4sW_s^2 + 2(t-s)^2 - 2s^2$$

$$\mathbb{E}(B_t \mid \mathcal{F}_s) = B_s + 4W_s^2(t-s) + 2(t-s)^2$$

da cui anche $(W_t^2 - t)^2 - B_t$ è una martingala! Giusto, ma B_t non è un processo a traiettorie crescenti come invece lo sono le traiettorie di

$$A_t = \int_0^t W_r^2\, dr$$