Lecture Notes in Integral Transforms

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Chapter 1
Fourier Series

Let $\mathbb{R}^\infty$ denote the space of $\mathbb{R}$-valued sequences. Sometimes the sequence $\alpha = \{\alpha_n\}_n$ in $\mathbb{R}^\infty$ will be written as

$$\alpha = (\alpha_1, \alpha_2, \alpha_3, ...) \tag{1.1}$$

Exercise 1.0.1 $\mathbb{R}^\infty$ *is a real vector space.*

Given a sequence α as in 1.1, following the approach of old books, the common practice is that of changing names and of constructing two new $\mathbb{R}$-valued sequences from α. So

$$a_0 := 2\alpha_1 \tag{1.2}$$

$$\forall k \in \mathbb{N} \qquad a_k := \alpha_{2k} \tag{1.3}$$

$$b_k := \alpha_{2k+1} \tag{1.4}$$

Next step is the construction of a sequence $\{s_n\}_{n\in\mathbb{N}\cup\{0\}}$ of functions $\mathbb{R} \to \mathbb{R}$: if $x \in \mathbb{R}$

$$s_0 := \frac{a_0}{2} \tag{1.5}$$

$$\forall n \in \mathbb{N} \qquad s_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \tag{1.6}$$

Having a sequence, the natural question deals with convergence: does $\{s_n\}_{n\in\mathbb{N}\cup\{0\}}$ converge to something in some sense? For example

- $\{s_n\}_{n\in\mathbb{N}\cup\{0\}}$ might have the property of pointwise convergence, i.e. fixed $x \in \mathbb{R}$, the real-valued sequence $\{s_n(x)\}_{n\in\mathbb{N}\cup\{0\}}$ admits limit.
- $\{s_n\}_{n\in\mathbb{N}\cup\{0\}}$ might have the property of L^p convergence for some $1 \le p \le \infty$. In particular, it is interesting the property of L^2 convergence, since L^2 can be endowed with the structure of Hilbert space.

If $\{s_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges in some sense, it is customary to let s denote the $\mathbb{R} \to \mathbb{R}$ limit function and also to write for $x \in \mathbb{R}$

$$s(x) =: \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] \tag{1.7}$$

Theorem 1.0.1 *Let $\alpha \in \mathbb{R}^\infty$ satisfy $\sum_{k=1}^{\infty} |\alpha_k| < +\infty$. Then*

$$s_n \longrightarrow s \quad \textit{pointwise (uniformly)}$$

Proof. The pointwise convergence will be prove showing that for $x \in \mathbb{R}$ and $\forall p > 0$

$$|s_{n+p}(x) - s_n(x)| \overset{n\to\infty}{\longrightarrow} 0 \tag{1.8}$$

Indeed, the equivalence

$$\sum_{k=1}^{\infty} |\alpha_k| < +\infty \qquad \Leftrightarrow \qquad \sum_{k=1}^{\infty} |a_k| + |b_k| < +\infty \tag{1.9}$$

yields

$$\sum_{k=n+1}^{n+p} [|a_k| + |b_k|] \overset{n\to\infty}{\longrightarrow} 0 \tag{1.10}$$

Hence

$$|s_{n+p}(x) - s_n(x)| = \Big| \sum_{k=n+1}^{n+p} [a_k \cos(kx) + b_k \sin(kx)] \Big| \tag{1.11}$$

$$\leq \sum_{k=n+1}^{n+p} [|a_k||\cos(kx)| + |b_k||\sin(kx)|] \tag{1.12}$$

$$\leq \sum_{k=n+1}^{n+p} [|a_k| + |b_k|] \overset{n\to\infty}{\longrightarrow} 0 \tag{1.13}$$

The convergence is also uniform: since the above passages do not depend on x,or rather $\sum_{k=n+1}^{n+p} [|a_k| + |b_k|] \overset{n\to\infty}{\longrightarrow} 0$ no matter which x was chosen, it can be easily seen that the sequence satisfies the definition of uniform convergence.

Example 1.0.1

Let

$$\alpha := \left(0, 1, 0, \frac{1}{2^2}, 0, \frac{1}{3^2}, 0, ...\right)$$

Then

$$\sum_{k=1}^{\infty} |\alpha_k| = \sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \sum_{k=2}^{\infty} \frac{1}{k^2}$$
$$< 1 + \sum_{k=2}^{\infty} \frac{1}{(k-1)k} = 1 + \sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k}\right) = 2$$

Hence

$$\sum_{k=1}^{n}\frac{\cos(k(\cdot))}{k^2}\longrightarrow\sum_{k=1}^{\infty}\frac{\cos(k(\cdot))}{k^2}\quad\text{uniformly pointwise}$$

Similarly, considering

$$\alpha := \left(0\,,0,\,1\,,0\,,\,\frac{1}{2^2}\,,0\,,\,\frac{1}{3^2}\,,0\,,\,...\right)$$

$$\sum_{k=1}^{n}\frac{\sin(k(\cdot))}{k^2}\longrightarrow\sum_{k=1}^{\infty}\frac{\sin(k(\cdot))}{k^2}\quad\text{uniformly pointwise}$$

Exercise 1.0.2 *Choose some $n\in\mathbb{N}$ and plot $\sum_{k=1}^{n}\frac{\sin(k(\cdot))}{k^2}$ and $\sum_{k=1}^{n}\frac{\cos(k(\cdot))}{k^2}$.*

Example 1.0.2

The sequences

$$\alpha := (1,1,1,1,...)$$
$$\tilde{\alpha} := \left(1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...\right)$$

do not satisfy the condition of absolute convergence, whereas for a real $c>1$

$$\hat{\alpha} := \left(1,\frac{1}{2^c},\frac{1}{3^c},\frac{1}{4^c},...\right)$$

does, thus the sequence $\{s_n\}_{n\in\mathbb{N}\cup\{0\}}$ where

$$s_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\left[\frac{\cos(kx)}{(2k)^c}+\frac{\sin(kx)}{(2k+1)^c}\right]$$

converges uniformly pointwise to a function $s:\mathbb{R}\longrightarrow\mathbb{R}$ that is usually written as

$$s(x)=\frac{1}{2}+\sum_{k=1}^{\infty}\left[\frac{\cos(kx)}{(2k)^c}+\frac{\sin(kx)}{(2k+1)^c}\right]$$

Esercizio 1.0.3 *Define for $n\in\mathbb{N}$ $\quad\gamma_n := \left(\sum_{k=1}^{n}\frac{1}{k}\right)-\log(n)$. Then the sequence $\{\gamma_n\}_n$ converges.*

Hence the sequence $\{\gamma_n\}_n$ can be used to construct converging sequences of $\mathbb{R}\longrightarrow\mathbb{R}$ functions.
Let γ denote the limit of $\{\gamma_n\}_n$. γ is called Euler-Mascheroni constant. It is still not known whether it is rational or irrational.

A classical result of real analysis states that if a sequence of $\mathbb{R}\longrightarrow\mathbb{R}$ continuous functions converges uniformly, the limit is continuous. Hence, from Theorem 1.0.1 it follows

Corollary 1.0.1 *Let $\alpha \in \mathbb{R}^\infty$ satisfy $\sum_{k=1}^{\infty} |\alpha_k| < +\infty$. Then the limit s of the corrisponding sequence $\{s_n\}_n$ is a continuous function.*

Define

$$l_1 := \{\alpha \in \mathbb{R}^\infty : \sum_{k=1}^{\infty} |\alpha_k| < \infty\} \tag{1.14}$$

Exercise 1.0.4 $\alpha, \beta \in l_1,\ \lambda \in \mathbb{R} \quad \Rightarrow \quad \alpha + \lambda\beta \in l_1.$

The above exercise implies that l_1 is a real vector space. Now, for $\alpha \in l_1$ define a function $\|\cdot\| : l_1 \longrightarrow \mathbb{R}$ by

$$\|\alpha\| := \sum_{k=1}^{\infty} |\alpha_k| \tag{1.15}$$

This function is well defined thanks to the definition of l_1.

Exercise 1.0.5 *Prove that l_1 is a Banach space, i.e. that*

1. *$(l_1, \|\cdot\|)$ is a normed vector space.*
2. *$(l_1, \|\cdot\|)$ is complete, namely*

$$\{\alpha_n\}_n \text{ Cauchy sequence in } l_1 \quad \Rightarrow \quad \alpha_n \to \alpha \in l_1$$

Consider a real number $p \geq 1$. Define

$$l_p := \{\alpha \in \mathbb{R}^\infty : \sum_{k=1}^{\infty} |\alpha_k|^p < \infty\} \tag{1.16}$$

and define a function $\|\cdot\|_p : l_p \longrightarrow \mathbb{R}$ by

$$\|\alpha\|_p := \sum_{k=1}^{\infty} |\alpha_k|^p \tag{1.17}$$

Exercise 1.0.6 *1. l_p is a real vector space.*
2. $(l_p, \|\cdot\|_p)$ is a normed space.
3. $(l_p, \|\cdot\|_p)$ is a Banach space.

Using the modern language of l_p spaces, Theorem 1.0.1 can be stated again as

$$\alpha \in l_1 \quad \Rightarrow \quad s_n \longrightarrow s \text{ pointwise (uniformly)} \tag{1.18}$$

Exercise 1.0.7 $p_1, p_2 \in \mathbb{R}$ *such that* $1 \leq p_1 < p_2 \ \Rightarrow \ l_{p_1} \subset l_{p_2}$

Solution of Exercise 1.0.7. Let $\alpha \in l_{p_1}$. Then, from the definition of Cauchy sequence, $\forall N \in \mathbb{N}\ \exists \varepsilon_N \in \mathbb{R}$ such that if $n, p \in \mathbb{N}$ and $n > N$ then

$$\sum_{j=n+1}^{n+p} |\alpha_k|^{p_1} < \varepsilon_N \tag{1.19}$$

To prove the statement of the exercise, one has to show that an analogous property holds when p_2 replaces p_1. Indeed, without loss of generality, one can assume that

$\forall k \in \mathbb{N}\ \alpha_k < 1$ (since the series converges this has to be true from a fixed index on). Then $\alpha_k^{p_1} > \alpha_k^{p_2}$, hence $\forall N \in \mathbb{N}\ \exists \varepsilon_N \in \mathbb{R}$ such that if $n, p \in \mathbb{N}$ and $n > N$

$$\sum_{j=n+1}^{n+p} |\alpha_k|^{p_2} < \sum_{j=n+1}^{n+p} |\alpha_k|^{p_1} < \varepsilon_N \tag{1.20}$$

and the statement is proved.

However, the converse does not hold: for example, if $p_1 = 1$ and $p_2 = 2$, let

$$\alpha := \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$$

then $\alpha \in l_{p_2}$, but $\alpha \notin l_{p_1}$. Hence $l_{p_1} \subsetneq l_{p_2}$.

Define a function $\|\cdot\|_\infty : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ by $\|\alpha\|_\infty := \sup_k |\alpha_k|$. The subset of $\mathbb{R}^\infty$ containing all bounded sequences is called

$$l_\infty := \{\alpha \in \mathbb{R}^\infty : \ \|\alpha\|_\infty < +\infty\} \tag{1.21}$$

Also, consider another subset of $\mathbb{R}^\infty$

$$c_0 := \{\alpha \in \mathbb{R}^\infty : \ \exists \lim_{k\to\infty} \alpha_k \text{ and } \lim_{k\to\infty} \alpha_k = 0\} \tag{1.22}$$

Exercise 1.0.8 *1.* l_∞, c_0 *are real vector spaces.*
2. $(l_\infty, \|\cdot\|_\infty), (c_0, \|\cdot\|_\infty)$ *are normed spaces.*
3. $(l_\infty, \|\cdot\|_\infty), (c_0, \|\cdot\|_\infty)$ *are Banach spaces.*
4. $\forall p \in \mathbb{R}$ *such that* $p > 1 \qquad l_p \subset c_0 \subset l_\infty$.

Let

$$C([0,2\pi]) := \{f \in \mathbb{R}^{[0,2\pi]} : \ f \text{ is continuous}\} \tag{1.23}$$

Define a function $\|\cdot\| : \mathbb{R}^{[0,2\pi]} \longrightarrow \mathbb{R} \cup \{+\infty\}$ as

$$\|f\| := \sup_{x\in[0,2\pi]} |f(x)| \tag{1.24}$$

Exercise 1.0.9
$C([0,2\pi])$ *is a real vector space.*
$(C([0,2\pi]), \|\cdot\|)$ *is a normed space.*
$(C([0,2\pi]), \|\cdot\|)$ *is a Banach space.*

Restricting the domain of s_n functions to $[0,2\pi]$, Theorem 1.0.1 states

Theorem 1.0.2 *If* $\alpha \in l_1 \quad \Rightarrow \quad \exists s \in C([0,2\pi])$ *such that* $s_n \longrightarrow s$ *pointwise (uniformly).*

Thus it is defined a function $l_1 \longrightarrow C([0,2\pi])$ that sends a sequence α to a continuous function s.

Exercise 1.0.10 *Check the orthogonality relations for sine and cosine functions, i.e. verify that for $n,m \in \mathbb{N} \cup \{0\}$*

$$\int_0^{2\pi} \cos(nx)\cos(mx)\mathrm{d}x = \begin{cases} 0 & \textit{if } n \neq m \\ \pi & \textit{if } n = m > 0 \\ 2\pi & \textit{if } n = m = 0 \end{cases} \tag{1.25}$$

$$\int_0^{2\pi} \sin(nx)\sin(mx)\mathrm{d}x = \begin{cases} 0 & \textit{if } n \neq m \textit{ or } n = m = 0 \\ \pi & \textit{if } n = m > 0 \end{cases} \tag{1.26}$$

$$\int_0^{2\pi} \cos(nx)\sin(mx)\mathrm{d}x = 0 \tag{1.27}$$

Let $\alpha \in \mathbb{R}^\infty$. Thanks to the previous exercise, for $m \in \mathbb{N} \cup \{0\}$ it is easy to compute the following integrals:

$$\begin{aligned} \int_0^{2\pi} \cos(mx)s_n(x)\mathrm{d}x &= \frac{a_0}{2}\int_0^{2\pi} \cos(mx)\mathrm{d}x + \sum_{k=1}^{n} a_k \int_0^{2\pi} \cos(mx)\cos(kx)\mathrm{d}x + \\ &\quad + \sum_{k=1}^{n} b_k \int_0^{2\pi} \cos(mx)\sin(kx)\mathrm{d}x \\ &= \begin{cases} 0 & \text{if } n < m \\ \pi a_m & \text{if } n \geq m \end{cases} \end{aligned} \tag{1.28}$$

$$\int_0^{2\pi} \sin(mx)s_n(x)\mathrm{d}x = \begin{cases} 0 & \text{if } n < m \\ \pi b_m & \text{if } n \geq m \end{cases} \tag{1.29}$$

This way, for a fixed m, define

$$A_n := \frac{1}{\pi}\int_0^{2\pi} \cos(mx)s_n(x)\mathrm{d}x = \begin{cases} 0 & \text{if } n < m \\ a_m & \text{if } n \geq m \end{cases} \tag{1.30}$$

$$B_n := \frac{1}{\pi}\int_0^{2\pi} \sin(mx)s_n(x)\mathrm{d}x = \begin{cases} 0 & \text{if } n < m \\ b_m & \text{if } n \geq m \end{cases} \tag{1.31}$$

Theorem 1.0.2 holds for the sequence $\{s_n\}_n$ and trigonometric functions are bounded in absolute value by 1, it follows that $\cos(m\cdot)s_n(\cdot) \longrightarrow \cos(m\cdot)s(\cdot)$ and $\sin(m\cdot)s_n(\cdot) \longrightarrow \sin(m\cdot)s(\cdot)$ uniformly. Moreover the intervals of integration above are bounded, hence for $n \to \infty$ Riemann integral allows the following passages:

$$\begin{aligned} a_m &= \lim_{n\to\infty} a_m = \lim_{n\to\infty} \frac{1}{\pi}\int_0^{2\pi} \cos(mx)s_n(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{2\pi} \lim_{n\to\infty}(\cos(mx)s_n(x))\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\cos(mx)s(x)\mathrm{d}x \end{aligned} \tag{1.32}$$

$$\begin{aligned} b_m &= \lim_{n\to\infty} b_m = \lim_{n\to\infty} \frac{1}{\pi}\int_0^{2\pi} \sin(mx)s_n(x)\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{2\pi} \lim_{n\to\infty}(\sin(mx)s_n(x))\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\sin(mx)s(x)\mathrm{d}x \end{aligned} \tag{1.33}$$

Is it possible to invert the above procedure? I.e., let $f \in L^1(0,2\pi)$. Then, since sine and cosine are bounded, the product of f and a trigonometric function still belongs to $L^1(0,2\pi)$, so it is feasible to associate to f a sequence α in $\mathbb{R}^\infty$ where

$$a_m := \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\mathrm{d}x \tag{1.34}$$

$$b_m := \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)\mathrm{d}x \tag{1.35}$$

$$\alpha := \left(\frac{a_0}{2}, a_1, b_1, a_2, b_2, a_3, b_3, \ldots\right) \tag{1.36}$$

The entries of the sequence α in 1.36 are called Fourier coefficients of f. Next, one can associate to α the sequence $\{s_n\}_n$. The idea of "inverting"the above passages can be concretized wondering:

1. Does $\{s_n\}_n$ converge pointwise?
2. If $\{s_n\}_n$ converges pointwise, is the limit f?

In general, the answers are *no*. Indeed, Kolmogorov proved the existence of $f \in L^1(0,2\pi)$ such that the sequences of $\{s_n(x)\}_n$ are divergent for every x.
Nevetheless, from previous theorems, if $\alpha \in l_1$, then exists a continuous limit s. Since s is continuous, $s \in L^1(0,2\pi)$ ($C([0,2\pi]) \subset L^1(0,2\pi)$).

Exercise 1.0.11 *$p, p' \in \mathbb{R}$ such that $1 \le p < p' \quad \Rightarrow \quad L^{p'}(0,2\pi) \subset L^p(0,2\pi)$. Moreover, $C([0,2\pi]) \subset L^p([0,2\pi])$.*

solution of Exercise 1.0.11. Let $f \in L^{p'}(0,2\pi)$. Actually, in the following argument, f is a representative of a class, not a class in $L^{p'}(0,2\pi)$. Then

$$\begin{aligned} \int_0^{2\pi} |f(x)|^p \mathrm{d}x &= \int_{|f|\le 1} |f(x)|^p \mathrm{d}x + \int_{|f|>1} |f(x)|^p \mathrm{d}x \\ &\le \int_0^{2\pi} 1\mathrm{d}x + \int_{|f|>1} |f(x)|^{p'}\mathrm{d}x \le 2\pi + \int_0^{2\pi} |f(x)|^{p'}\mathrm{d}x \\ &= 2\pi + \|f\|_{L^{p'}} \in \mathbb{R} \end{aligned}$$

Moreover, if $f \in C([0,2\pi])$, $|f|^p \in C([0,2\pi])$, therefore, using Weierstrass theorem

$$\|f\|_{L^p} = \int_0^{2\pi} |f(x)|^p \mathrm{d}x \le \int_0^{2\pi} \max_{t\in[0,2\pi]} (|f(t)|^p)\mathrm{d}x = 2\pi \max_{t\in[0,2\pi]} (|f(t)|^p) \in \mathbb{R}$$

One may now cast a glance at what happens if $f \in L^2(0,2\pi) \subset L^1(0,2\pi)$.
Notice that, since s_n are continuous, they belong to $L^2(0,2\pi)$. Next result deals with convergence of $\{s_n(x)\}_n$ when the sequence comes from $\alpha \in l_2$. Actually, a stronger convergence than the one stated in the theorem holds.

Theorem 1.0.3 *Let $\alpha \in l_2$. Then $\exists s \in L^2(0,2\pi)$ such that $s_n \xrightarrow{L^2} s$.*

Proof. Since $(L^2(0,2\pi), \|\cdot\|_{L^2})$ is a Banach space, it is complete, hence it suffices to prove that $\{s_n(x)\}_n$ is a Cauchy sequence in $(L^2(0,2\pi), \|\cdot\|)$, i.e. $\forall p \in \mathbb{N}, p > 0$ $\|s_{n+p} - s_n\|_{L^2} \xrightarrow{n\to\infty} 0$.
In the computation, the equality labelled with † is achieved thanks to the linearity of the integral, 1.25, 1.26 and 1.27, whereas the one labelled with ‡ follows from 1.25 and 1.26:

$$\begin{aligned}
\|s_{n+p} - s_n\|_{L^2}^2 &= \int_0^{2\pi} (s_{n+p}(x) - s_n(x))^2 \mathrm{d}x \\
&= \int_0^{2\pi} \Big(\sum_{k=n+1}^{n+p} a_k \cos(kx) + b_k \sin(kx) \Big)^2 \mathrm{d}x \\
&\overset{\dagger}{=} \sum_{k=n+1}^{n+p} \int_0^{2\pi} a_k^2 \cos^2(kx)\mathrm{d}x + \sum_{k=n+1}^{n+p} \int_0^{2\pi} b_k^2 \sin^2(kx)\mathrm{d}x \\
&\overset{\ddagger}{=} \pi \sum_{k=n+1}^{n+p} (a_k^2 + b_k^2) \xrightarrow{n\to\infty} 0 \qquad (1.37)
\end{aligned}$$

where the limit ensues from $\alpha \in l_2$: $\sum_k \alpha_k^2 < \infty$ implies $\sum_{i=j+1}^{j+p} \alpha_i \xrightarrow{j\to\infty} 0$ and the corresponding expression written in terms of a_k, b_k.

Statement 1 *Let $f \in L^2(0,2\pi)$. Then the sequence α associated to f in 1.36 belongs to l_2.*

(The above statement will be proved in Theorem 1.0.6) In this situation the answers to the foregoing questions are not completely negative:

1. Theorem 1.0.3 implies that $\{s_n\}_n$ converges (in L^2 sense) to an $s \in L^2(0,2\pi)$. Actually, the convergence is stronger: it is an a.e. convergence (Conjecture 1.0.1).
2. It can be proved that $f = s$ a.e. (Conjecture 1.0.1).

Conjecture 1.0.1 (Lusin, 1915) *If $f \in L^2(0,2\pi)$ then $s_n \longrightarrow f$ a.e..*

In 1966 Lennart Carleson proved Lusin's Conjecture (L. Carleson: *On convergence and growth of partial sums of Fourier series*. Acta Math. 116, 135-157 (1966)). Till that moment, mathematicians had thought the Conjecture to be false. In 1973 Charles Fefferman gave a different proof of the Conjecture.
Then, in 1967, a generalization was proved:

Theorem 1.0.4 (Hunt, 1967) *If $p \in \mathbb{R}$, $p > 1$ and $f \in L^p$ then $s_n \longrightarrow f$ a.e..*

Thanks to Kolmogorov's example, the above theorem does not hold if $p = 1$.

Exercise 1.0.12 *i. If $f, g \in L^1(0, 2\pi)$, then if $\alpha(f), \alpha(g), \alpha(f+g)$ are the series associated as in 1.36 to f, g and $f + g (\in L^1(0, 2\pi))$ respectively, then $\forall m \in \mathbb{N}$ $\alpha_m(f+g) = \alpha_m(f) + \alpha_m(g)$.*
ii. If $f \in L^2(0, 2\pi)$ and $s \in L^2(0, 2\pi)$ is the limit of the sequence $\{s_n\}_n$, then $\forall m \in \mathbb{N}$ $\alpha_m(f) = \alpha_m(s)$, equivalently, $\alpha_m(f - s) = 0$.

If $z \in \mathbb{C}$, two possible definitions of e^z are, provided that the existence of the limits has been proved,

$$e^z := \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n \tag{1.38}$$

$$e^z := \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{1.39}$$

Exercise 1.0.13 *i. (Euler's Identity) $\forall x \in \mathbb{R}$ $\quad e^{ix} = \cos x + i \sin x$.*
ii. $z_1, z_2 \in \mathbb{C}$ $\quad e^{z_1} e^{z_2} = e^{z_1 + z_2}$.

Hence, for $x \in \mathbb{R}$, since

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \quad \text{and} \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{1.40}$$

$$\begin{aligned}
s_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx) \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} a_k \frac{e^{ikx} + e^{-ikx}}{2} + \sum_{k=1}^{n} b_k \frac{e^{ikx} - e^{-ikx}}{2i} \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} \frac{ia_k + b_k}{2i} e^{ikx} + \sum_{k=1}^{n} \frac{ia_k - b_k}{2i} e^{-ikx} \\
&= \frac{a_0}{2} + \sum_{k=1}^{n} \frac{a_k - ib_k}{2} e^{ikx} + \sum_{k=-n}^{-1} \frac{a_{-k} + ib_{-k}}{2} e^{ikx} \\
&= \frac{1}{2} \sum_{k=-n}^{n} c_k e^{ikx}
\end{aligned} \tag{1.41}$$

where

$$c_k := \begin{cases} a_k - ib_k & \text{if} \quad 0 < k \le n \\ a_0 & \text{if} \quad k = 0 \\ a_{-k} + ib_{-k} & \text{if} \quad -n \le k < 0 \end{cases}$$

Lemma 1.0.1 *Let $\beta, r \in \mathbb{R}$, then*

$$\cos(r\beta)\cos(\beta) = \frac{\cos\big((r+1)\beta\big) + \cos\big((r-1)\beta\big)}{2} \tag{1.42}$$

$$\sin(r\beta)\sin(\beta) = \frac{\cos\big((r-1)\beta\big) - \cos\big((r+1)\beta\big)}{2} \tag{1.43}$$

$$\sin(r\beta)\cos(\beta) = \frac{\sin\big((r+1)\beta\big) + \sin\big((r-1)\beta\big)}{2} \tag{1.44}$$

$$\cos(r\beta)\sin(\beta) = \frac{\sin\big((r+1)\beta\big) - \sin\big((r-1)\beta\big)}{2} \tag{1.45}$$

Proof. Thanks to $\cos(-\beta) = \cos(\beta)$ and $\sin(-\beta) = -\sin(\beta)$, one gets

$$\begin{cases} \cos(r\beta)\cos(\beta) - \sin(r\beta)\sin(\beta) = \cos\big((r+1)\beta\big) \\ \cos(r\beta)\cos(\beta) + \sin(r\beta)\sin(\beta) = \cos\big((r-1)\beta\big) \end{cases}$$

$$\begin{cases} \sin(r\beta)\cos(\beta) + \cos(r\beta)\sin(\beta) = \sin\big((r+1)\beta\big) \\ \sin(r\beta)\cos(\beta) - \cos(r\beta)\sin(\beta) = \sin\big((r-1)\beta\big) \end{cases}$$

Combining in each system the two equations, once adding them, once subtracting, one gets the equalities displayed above.

Theorem 1.0.5 *Let $f \in L^1(0,2\pi)$ and suppose all Fourier coefficients of f to be zero. Then $f = 0$ a.e..*

Proof. `-First step. Let` $f \in C([0,2\pi]) \subset L^1(0,2\pi)$ `and suppose all Fourier coefficients of` f `to be zero. Then` $f \equiv 0$ `.`
By contradiction, suppose $f \neq 0$. Then, since $f \in C([0,2\pi])$, $\exists c \in \mathbb{R}, t_0 \in]0,2\pi[, \delta \in \mathbb{R}$ with $\delta > 0$ such that $[t_0-\delta, t_0+\delta] \subset]0,2\pi[$ and $\forall t \in [t_0-\delta, t_0+\delta]$ $f(y) \geq c > 0$ (without loss of generality it is possible to assume $f(t_0) > 0$, in fact otherwise it suffices to consider $-f$ since, thanks to linearity, also the Fourier coefficients of $-f$ are zero.). Define for $n \in \mathbb{N}$ and $t \in [0,2\pi]$

$$\psi_n(t) := \Big(\cos(t-t_0) + 1 - \cos\delta\Big)^n$$

If $t \in [t_0-\delta, t_0+\delta]$, then $0 \leq |t-t_0| \leq \delta$, and thus $\cos(t-t_0)+1-\cos\delta \geq \cos\delta + 1 - \cos\delta = 1$, so $\psi_n(t) \geq 1$; analogously, if $t \in [0,2\pi] \setminus [t_0-\delta, t_0+\delta]$, $\psi_n(t) \leq 1$. Moreover $\exists \delta' \in \mathbb{R}$, $\delta > \delta' > 0$ such that $\exists \rho \in \mathbb{R}$, $\rho > 0$ such that $\cos(t-t_0)+1-\cos\delta \leq 1+\rho$ if $t \in [t_0-\delta', t_0+\delta']$. Thus

$$\int_0^{2\pi} f(t)\psi_n(t)\mathrm{d}t \overset{n\to\infty}{\longrightarrow} +\infty \tag{1.46}$$

Indeed

$$\int_0^{2\pi} f(t)\psi_n(t)\mathrm{d}t = \overbrace{\int_{t_0-\delta}^{t_0+\delta} f(t)\psi_n(t)\mathrm{d}t}^{A} + \overbrace{\int_{[0,2\pi]\setminus[t_0-\delta,t_0+\delta]} f(t)\psi_n(t)\mathrm{d}t}^{B}$$

and B is finite because, if $M \in \mathbb{R}$ is such that $\forall t \in [0, 2\pi] \quad |f(t)| \le M$,

$$|B| \le \int_{[0,2\pi] \setminus [t_0-\delta, t_0+\delta]} |f(t)\psi_n(t)| \mathrm{d}t \le \int_{[0,2\pi] \setminus [t_0-\delta, t_0+\delta]} M \cdot 1 \mathrm{d}t \le 2\pi M$$

whereas

$$A \ge \int_{t_0-\delta}^{t_0+\delta} c\psi_n(t) \mathrm{d}t \ge c \int_{t_0-\delta'}^{t_0+\delta'} \psi_n(t) \mathrm{d}t \ge c \int_{t_0-\delta'}^{t_0+\delta'} (1+\rho)^n \mathrm{d}t \ge 2c\delta'(1+\rho)^n \overset{n\to\infty}{\longrightarrow} +\infty$$

A contradiction arises showing that from the hypotheses it follows that $\forall n \in \mathbb{N}$ $\int_0^{2\pi} f(t)\psi_n(t) \mathrm{d}t = 0$, since this implies $\lim_{n\to\infty} \int_0^{2\pi} f(t)\psi_n(t) \mathrm{d}t = 0$.
First, by induction one proves that $\forall n \in \mathbb{N}\ \exists A_k, B_k \in \mathbb{R}$ such that

$$\psi_n(t) = A_0 + \sum_{k=1}^{n} \Big[A_k \cos(kt) + B_k \sin(kt) \Big] \tag{1.47}$$

Indeed, if $n = 1$

$$\psi_1(t) = \cos(t - t_0) + 1 - \cos\delta = \overbrace{1 - \cos\delta}^{A_0} + \overbrace{\cos t_0}^{A_1} \cos t + \overbrace{(-\sin t_0)}^{B_1} \sin t$$

Now, suppose 1.47 holds if $n \le N-1$, $N \in \mathbb{N}$, $N \ne 1, 2$. Then

$$\begin{aligned}
\psi_N(t) &= (\cos(t-t_0) + 1 - \cos\delta)\psi_{N-1}(t) \\
&= (\cos(t-t_0) + 1 - \cos\delta)\left(A_0 + \sum_{k=1}^{N-1} \Big[A_k \cos(kt) + B_k \sin(kt) \Big] \right) \\
&= \Big((1-\cos\delta) + \cos t_0 \cos t - \sin t_0 \sin t \Big) \left(A_0 + \sum_{k=1}^{N-1} \Big[A_k \cos(kt) + B_k \sin(kt) \Big] \right) \\
&= A_0(1-\cos\delta) + \sum_{k=1}^{N-1} A_k (1-\cos\delta)\cos(kt) + \sum_{k=1}^{N-1} B_k (1-\cos\delta)\sin(kt) + \\
&\qquad + A_0 \cos t_0 \cos t + \sum_{k=1}^{N-1} A_k \cos t_0 \cos t \cos(kt) + \sum_{k=1}^{N-1} B_k \cos t_0 \cos t \sin(kt) + \\
&\qquad - A_0 \sin t_0 \sin t - \sum_{k=1}^{N-1} A_k \sin t_0 \sin t \cos(kt) - \sum_{k=1}^{N-1} B_k \sin t_0 \sin t \sin(kt)
\end{aligned}$$

and, thanks to Lemma 1.0.1

$$
\begin{aligned}
&= A_0(1-\cos\delta) + \sum_{k=1}^{N-1} A_k(1-\cos\delta)\cos(kt) + \sum_{k=1}^{N-1} B_k(1-\cos\delta)\sin(kt) + \\
&\quad + A_0\cos t_0 \cos t + \sum_{k=1}^{N-1} \frac{A_k\cos t_0}{2}\Big(\cos\big((k+1)t\big) + \cos\big((k-1)t\big)\Big) + \\
&\qquad + \sum_{k=1}^{N-1} \frac{B_k\cos t_0}{2}\Big(\sin\big((k+1)t\big) + \sin\big((k-1)t\big)\Big) + \\
&\quad - A_0\sin t_0 \sin t - \sum_{k=1}^{N-1} \frac{A_k\sin t_0}{2}\Big(\sin\big((k+1)t\big) - \sin\big((k-1)t\big)\Big) + \\
&\qquad - \sum_{k=1}^{N-1} \frac{B_k\sin t_0}{2}\Big(\cos\big((k-1)t\big) - \cos\big((k+1)t\big)\Big) \\
&= A_0(1-\cos\delta) + \sum_{k=1}^{N-1} A_k(1-\cos\delta)\cos(kt) + \sum_{k=1}^{N-1} B_k(1-\cos\delta)\sin(kt) + \\
&\quad + A_0\cos t_0 \cos t + \sum_{k=2}^{N} \frac{A_{k-1}\cos t_0}{2}\cos(kt) + \sum_{k=0}^{N-2} \frac{A_{k+1}\cos t_0}{2}\cos(kt) + \\
&\qquad + \sum_{k=2}^{N} \frac{B_{k-1}\cos t_0}{2}\sin(kt) + \sum_{k=0}^{N-2} \frac{B_{k+1}\cos t_0}{2}\sin(kt) + \\
&\quad - A_0\sin t_0 \sin t - \sum_{k=2}^{N} \frac{A_{k-1}\sin t_0}{2}\sin(kt) + \sum_{k=0}^{N-2} \frac{A_{k+1}\sin t_0}{2}\sin(kt) + \\
&\qquad - \sum_{k=0}^{N-2} \frac{B_{k+1}\sin t_0}{2}\cos(kt) + \sum_{k=2}^{N} \frac{B_{k-1}\sin t_0}{2}\cos(kt) \\
&= A_0(1-\cos\delta) + \frac{A_1\cos t_0}{2} + \frac{B_1\sin t_0}{2} + \\
&\quad + \sum_{k=2}^{N-2}\left[A_k(1-\cos\delta) + \frac{A_{k-1}\cos t_0}{2} + \frac{A_{k+1}\cos t_0}{2} - \frac{B_{k+1}\sin t_0}{2} - \frac{B_{k-1}\sin t_0}{2}\right]\cos(kt) + \\
&\qquad + \left[A_1(1-\cos\delta) + A_0\cos t_0 + \frac{A_2\cos t_0}{2} - \frac{B_2\sin t_0}{2}\right]\cos(t) + \\
&\qquad + \left[A_{N-1}(1-\cos\delta) + \frac{A_{N-2}\cos t_0}{2} - \frac{B_{N-2}\sin t_0}{2}\right]\cos((N-1)t) + \\
&\qquad + \left[\frac{A_{N-1}\cos t_0}{2} - \frac{B_{N-1}\sin t_0}{2}\right]\cos(Nt) \\
&\quad + \sum_{k=2}^{N-2}\left[B_k(1-\cos\delta) + \frac{B_{k-1}\cos t_0}{2} + \frac{B_{k+1}\cos t_0}{2} - \frac{A_{k+1}\sin t_0}{2} - \frac{A_{k-1}\sin t_0}{2}\right]\sin(kt) + \\
&\qquad + \left[B_1(1-\cos\delta) - A_0\sin t_0 + \frac{B_2\cos t_0}{2} - \frac{A_2\sin t_0}{2}\right]\sin(t) + \\
&\qquad + \left[B_{N-1}(1-\cos\delta) + \frac{B_{N-2}\cos t_0}{2} - \frac{A_{N-2}\sin t_0}{2}\right]\sin((N-1)t) + \\
&\qquad + \left[\frac{B_{N-1}\cos t_0}{2} - \frac{A_{N-1}\sin t_0}{2}\right]\sin(Nt)
\end{aligned}
$$

Hence 1.47 is proved.
It follows that, by linearity, the condition on the Fourier coefficients of f yields

$$\forall n \in \mathbb{N} \qquad \int_0^{2\pi} f(t)\psi_n(t)\mathrm{d}t = 0$$

which contradicts 1.46. Thus $f \equiv 0$.
`-Second step. In the hypotheses of the theorem,` $f = 0$ `a.e.`
Define

$$F(t) := \int_0^t f(s)\mathrm{d}s$$

F is a continuous function, $\exists F'(t) = f(t)$ a.e. $t \in [0, 2\pi]$, $F(0) = 0$ and $F(2\pi) = \pi a_0 = 0$. Let A_k and B_k be Fourier coefficients of F. Then, for $k \geq 1$, (in these hypotheses, thanks to approximation results, it is possible to integrate by parts)

$$\begin{aligned}
\pi A_k &= \int_0^{2\pi} F(\xi)\cos(k\xi)\mathrm{d}\xi = \left[F(t)\frac{\sin(kt)}{k}\right]_0^{2\pi} - \frac{1}{k}\int_0^{2\pi} F'(\xi)\sin(k\xi)\mathrm{d}\xi \\
&= -\frac{1}{k}\int_0^{2\pi} f(\xi)\sin(k\xi)\mathrm{d}\xi = -\frac{\pi b_k}{k} = 0 \\
\pi B_k &= \int_0^{2\pi} F(\xi)\sin(k\xi)\mathrm{d}\xi = \left[-F(t)\frac{\cos(kt)}{k}\right]_0^{2\pi} + \frac{1}{k}\int_0^{2\pi} F'(\xi)\cos(k\xi)\mathrm{d}\xi \\
&= \frac{1}{k}\int_0^{2\pi} f(\xi)\cos(k\xi)\mathrm{d}\xi = -\frac{\pi a_k}{k} = 0
\end{aligned}$$

Define for $t \in [0, 2\pi]$ $\widetilde{F}(t) := F(t) - A_0$. Let $\widetilde{A}_k, \widetilde{B}_k$ be Fourier coefficients of $\widetilde{F}$. By linearity, for $k > 0$,

$$\widetilde{A}_0 = A_0 - A_0 = 0, \qquad \widetilde{A}_k = A_k - 0 = A_k = 0, \qquad \widetilde{B}_k = B_k - 0 = B_k = 0$$

$\widetilde{F}$ is continuous, hence $\widetilde{F}$ satisfies the hypotheses of the first step, therefore $\widetilde{F} \equiv 0$, i.e. $F \equiv A_0$. It follows that $F' \equiv 0$, and since $F' = f$ a.e., $f = 0$ a.e..

Definition 1.0.1 *A Hilbert space is a couple $(H, \langle\cdot,\cdot\rangle)$ such that H is a vector space over $\mathbb{R}$ or $\mathbb{C}$, $\langle\cdot,\cdot\rangle : H \times H \longrightarrow \mathbb{C}$ is an inner product and, if $\|\cdot\|$ denotes the norm induced by $\langle\cdot,\cdot\rangle$, $(H, \|\cdot\|)$ is a Banach space.*

Recall that $L^2(0, 2\pi)$ with the inner product $\langle f, g\rangle := \int_0^{2\pi} f(t)g(t)\mathrm{d}t$ is a Hilbert space.

Theorem 1.0.6 *Let $f \in L^2(0, 2\pi)$. Let a_k, b_k be Fourier coefficients of f. Then*

$$\|s_n\|_{L^2} \leq \|f\|_{L^2} \tag{1.48}$$

and

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\|f\|_{L^2}^2 \tag{1.49}$$

In particular, if α is the sequence of Fourier coefficients of f, then $\alpha \in l_2$. 1.49 is called Bessel's inequality.

Proof. Thanks to 1.25, 1.26 and 1.27

$$\langle s_n, s_n \rangle = \int_0^{2\pi} s_n^2(t)\mathrm{d}t = \int_0^{2\pi} \left(\frac{a_0}{2} + \sum_{k=1}^{n} \Big[a_k \cos(kt) + b_k \sin(kt) \Big] \right)^2 \mathrm{d}t = \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right)$$

Thanks to linearity and to the definition of Fourier coefficients

$$\langle s_n, f \rangle = \int_0^{2\pi} s_n(t) f(t)\mathrm{d}t = \frac{a_0}{2} \int_0^{2\pi} f(t)\mathrm{d}t + \sum_{k=1}^{n} \left(a_k \int_0^{2\pi} f(t)\cos(kt)\mathrm{d}t + b_k \int_0^{2\pi} f(t)\sin(kt)\mathrm{d}t \right)$$
$$= \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right)$$

Therefore, also using the Cauchy-Schwarz inequality, it follows

$$\|s_n\|_{L^2}^2 = |\langle s_n, f \rangle| \le \|s_n\|_{L^2} \|f\|_{L^2}$$

Hence, for both $\|s_n\|_{L^2} = 0$ and $\|s_n\|_{L^2} \neq 0$, $\|s_n\|_{L^2} \le \|f\|_{L^2}$.
Moreover, $\forall n \in \mathbb{N}$

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \le \frac{1}{\pi} \|f\|_{L^2}^2 \in \mathbb{R}$$

Thus the sequence $\left\{ \frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right\}_n$ is a bounded and non-decreasing, hence it admits limit, i.e. $\alpha \in l_2$. Moreover the limit, denoted with $\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$, satisfies 1.49.

Remark 1 *Actually, if $f \in L^2(0, 2\pi)$, then $\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{\pi} \|f\|_{L^2}^2$.*

An application. Let $A, B \in \mathbb{R}$ and define $h(x) := Ax + B$ for $x \in [0, 2\pi]$. Since $h \in C([0, 2\pi])$, from Exercise 1.0.11 it follows immediately that $\forall p \in \mathbb{R}$, $p \ge 1$, $h \in L^p([0, 2\pi])$.

Exercise 1.0.14 *Let a_k, b_k be Fourier coefficients of h. Show that*

$$a_k = \begin{cases} 2(A\pi + B) & \text{if } k = 0 \\ 0 & \text{if } k \ge 1 \end{cases} \qquad b_k = -\frac{2}{k} A \tag{1.50}$$

Solution of Exercise 1.0.14. For $m \in \mathbb{N}$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}(Ax+B)\mathrm{d}x = \frac{1}{\pi}\left(A\left[\frac{x^2}{2}\right]_0^{2\pi} + B\big[x\big]_0^{2\pi}\right) = \frac{1}{\pi}(A2\pi^2 + B2\pi) = 2(A\pi + B)$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi}(Ax+B)\cos(mx)\mathrm{d}x = \frac{A}{\pi}\left(\overbrace{\left[\frac{x\sin(mx)}{m}\right]_0^{2\pi}}^{0} - \overbrace{\int_0^{2\pi}\sin(mx)\mathrm{d}x}^{0}\right) + \frac{B}{\pi}\overbrace{\int_0^{2\pi}\cos(mx)\mathrm{d}x}^{0} = 0$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi}(Ax+B)\sin(mx)\mathrm{d}x = \frac{A}{\pi}\left(\left[\frac{-x\cos(mx)}{m}\right]_0^{2\pi} + \overbrace{\int_0^{2\pi}\cos(mx)\mathrm{d}x}^{0}\right) + \frac{B}{\pi}\overbrace{\int_0^{2\pi}\sin(mx)\mathrm{d}x}^{0} = -\frac{2}{m}A$$

If A, B are such that $B = -A\pi$, then $\forall k \in \mathbb{N}\ a_k = 0$. In particular, if $A = -\frac{1}{2}$ and $B = \frac{\pi}{2}$, i.e. $h(x) = \frac{\pi - x}{2}$, then $a_0 = 0$ and $\forall k \in \mathbb{N}\ a_k = 0$, $b_k = \frac{1}{k}$. Then from Remark 1, since $h \in L^2(0, 2\pi)$, it follows

$$\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{\pi}\int_0^{2\pi}h^2(t)\mathrm{d}t = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-t)^2}{4}\mathrm{d}t = \frac{1}{4\pi}\left[\frac{(t-\pi)^3}{3}\right]_0^{2\pi}$$
$$= \frac{1}{12\pi}(\pi^3 - (-\pi)^3) = \frac{\pi^2}{6}$$

The result $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$ is due to Euler (but he found it in a different way).
The sequence α of Fourier coefficients of h is such that $\alpha \in \bigcap_{p>1} l_p$ and $\alpha \notin l_1$.
In this situation, $s_n(x) = \sum_{k=1}^{n}\frac{\sin(kx)}{k}$ and since Conjecture 1.0.1 holds

$$s_n \longrightarrow \frac{\pi - x}{2} \qquad \text{a.e.}$$

Actually, a stronger convergence holds: in $(0, 2\pi)$ the convergence is pointwise and for $\delta \in \mathbb{R}$, $0 < \delta < \pi$, the convergence in $[\delta, 2\pi - \delta]$ is uniform (Example 1.1.1).

Definition 1.0.2 *Let $(H, \langle\cdot,\cdot\rangle)$ be an Hilbert space. A collection of vectors $\{v_\gamma\}_{\gamma\in A}$ in H is said to be an orthonormal system if for $\gamma, \mu \in A$*

$$\langle v_\gamma, v_\mu\rangle = \begin{cases} 1 & \text{if } \gamma = \mu \\ 0 & \text{if } \gamma \neq \mu \end{cases} \tag{1.51}$$

The collection of vectors $\{v_\gamma\}_{\gamma\in A}$ is complete if, given any $y \in H \setminus \{0\}$ $\exists \mu \in A$ such that $\langle v_\mu, y\rangle \neq 0$.
A complete orthonormal system is called an orthonormal basis.

Define, for $k \in \mathbb{N}$, functions $[0, 2\pi] \longrightarrow \mathbb{R}$

$$u_0(x) := \frac{1}{\sqrt{2\pi}}, \qquad u_{2k}(x) := \frac{\cos(kx)}{\sqrt{\pi}} \qquad u_{2k+1}(x) := \frac{\sin(kx)}{\sqrt{\pi}} \tag{1.52}$$

These functions are obviously in $L^2(0,2\pi)$ and from 1.25, 1.26 and 1.27 it is immediate to see that for $i,j \in (\mathbb{N}\cup\{0\})\smallsetminus\{1\}$

$$\langle u_i, u_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.53}$$

Hence the collection of vectors of $L^2(0,2\pi)$ $\{u_i\}_{i\in\mathbb{N}\smallsetminus\{1\}} \cup \{u_0\}$ is an orthonormal system in the Hilbert space $(L^2(0,2\pi), \langle\cdot,\cdot\rangle)$. Moreover, if $f \in L^2(0,2\pi)$ and all Fourier coefficients of f are 0, applying Theorem 1.0.5 (from Exercise 1.0.11 $L^2(0,2\pi)\subset L^1(0,2\pi)$), it follows that $f = 0$ a.e., i.e $f = 0$ in $L^2(0,2\pi)$. Because of the definitions, it is trivial to see that the condition "all the Fourier coefficients of f are 0 "is equivalent to the orthogonality of f to all the vectors of $\{u_i\}_{i\in\mathbb{N}\smallsetminus\{1\}} \cup \{u_0\}$. Therefore $\{u_i\}_{i\in\mathbb{N}\smallsetminus\{1\}} \cup \{u_0\}$ is also complete, hence it is an orthonormal basis.

Exercise 1.0.15 *Prove the extension of the Pythagorean Theorem in $L^2(0,2\pi)$, i.e. that if $f \in L^2(0,2\pi)$ then $\|f\|^2_{L^2} = \langle f,u_0\rangle^2 + \sum_{i=2}^{\infty}\langle f,u_k\rangle^2$.*

Let $f \in L^1(0,2\pi)$. Thanks to the definition of Fourier coefficients of f and thanks to the linearity and homogeneity of the integral

$$\begin{aligned} s_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n}\Big[a_k\cos(kx) + b_k\sin(kx)\Big] \\ &= \frac{1}{\pi}\int_0^{2\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\Big[\cos(kt)\cos(kx) + \sin(kt)\sin(kx)\Big]\right)\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^{2\pi} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos\big(k(x-t)\big)\right)\mathrm{d}t \end{aligned}$$

Define, for $n \in \mathbb{N}$, and $\vartheta \in \mathbb{R}$

$$D_n(\vartheta) := \frac{1}{2} + \sum_{k=1}^{n}\cos(k\vartheta) \stackrel{\dagger}{=} \frac{\sin\left(\left(n+\frac{1}{2}\right)\vartheta\right)}{2\sin\frac{\vartheta}{2}} \tag{1.54}$$

The function D_n is called Dirichlet kernel.

Exercise 1.0.16 *Prove † in 1.54.*
Also plot the function of ϑ $\frac{\sin\left(\left(n+\frac{1}{2}\right)\vartheta\right)}{2\sin\frac{\vartheta}{2}}$ for some n.

Hint to prove Exercise 1.0.16: Multiply the term on the left by $2\sin\frac{\vartheta}{2}$.

Hint to prove Exercise 1.0.16 in a different way: Consider $\mathfrak{Re}\left(\frac{1}{2} + \sum_{k=1}^{n} e^{ik\vartheta}\right)$.

Solution of Exercise 1.0.16 following the second hint. Notice that

$$\sum_{k=0}^{m-1}\cos(kx)+i\sum_{k=1}^{m-1}\sin(kx)=\sum_{k=0}^{m-1}e^{ikx}=\frac{1-e^{mx}}{1-e^{ix}}=\frac{1-\cos(mx)-i\sin(mx)}{1-\cos(x)-i\sin(x)}$$
$$=\frac{2\sin^2\frac{mx}{2}-i2\sin\frac{mx}{2}\cos\frac{mx}{2}}{2\sin^2\frac{x}{2}-i2\sin\frac{x}{2}\cos\frac{x}{2}}=\frac{\sin\frac{mx}{2}}{\sin\frac{x}{2}}\frac{\sin\frac{mx}{2}-i\cos\frac{mx}{2}}{\sin\frac{x}{2}-i\cos\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}+i\cos\frac{x}{2}}{\sin\frac{x}{2}+i\cos\frac{x}{2}}$$
$$=\frac{\sin^2\frac{mx}{2}}{\sin^2\frac{x}{2}}\;\frac{\sin\frac{mx}{2}\sin\frac{x}{2}+\cos\frac{mx}{2}\cos\frac{x}{2}+i\Big(\sin\frac{mx}{2}\cos\frac{x}{2}-\cos\frac{mx}{2}\sin\frac{x}{2}\Big)}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}$$
$$=\frac{\sin\frac{mx}{2}}{\sin\frac{x}{2}}\left(\cos\frac{(m-1)x}{2}+i\sin\frac{(m-1)x}{2}\right)$$

Hence, considering the real part and the imaginary part separately, one gets

$$\sum_{k=0}^{m-1}\cos(kx)=\frac{\sin\frac{mx}{2}}{\sin\frac{x}{2}}\;\cos\frac{(m-1)x}{2} \tag{1.55}$$

$$\sum_{k=1}^{m-1}\sin(kx)=\frac{\sin\frac{mx}{2}}{\sin\frac{x}{2}}\sin\frac{(m-1)x}{2} \tag{1.56}$$

Therefore

$$D_n(\vartheta)=\frac{1}{2}+\sum_{k=1}^{n}\cos(k\vartheta)=\frac{1}{2}+\sum_{k=0}^{n}\cos(k\vartheta)-1=-\frac{1}{2}+\frac{\sin\frac{(n+1)\vartheta}{2}}{\sin\frac{\vartheta}{2}}\;\cos\frac{n\vartheta}{2}$$
$$=\frac{2\sin\frac{(n+1)\vartheta}{2}\cos\frac{n\vartheta}{2}-\sin\frac{\vartheta}{2}}{2\sin\frac{\vartheta}{2}}=\frac{2\Big(\cos\frac{n\vartheta}{2}\sin\frac{\vartheta}{2}+\sin\frac{n\vartheta}{2}\cos\frac{\vartheta}{2}\Big)\cos\frac{n\vartheta}{2}-\sin\frac{\vartheta}{2}}{2\sin\frac{\vartheta}{2}}$$
$$=\frac{\Big(2\cos^2\frac{n\vartheta}{2}-1\Big)\sin\frac{\vartheta}{2}+2\sin\frac{n\vartheta}{2}\cos\frac{n\vartheta}{2}\cos\frac{\vartheta}{2}}{2\sin\frac{\vartheta}{2}}=\frac{\cos(n\vartheta)\sin\frac{\vartheta}{2}+\sin(n\vartheta)\cos\frac{\vartheta}{2}}{2\sin\frac{\vartheta}{2}}$$
$$=\frac{\sin\Big(\big(n+\frac{1}{2}\big)\vartheta\Big)}{2\sin\frac{\vartheta}{2}}$$

Hence
$$s_n(x)=\frac{1}{\pi}\int_0^{2\pi}f(t)D_n(x-t)\mathrm{d}t=\frac{1}{\pi}(f*D_n)(x) \tag{1.57}$$

($*$ denotes the convolution product). Notice that $|D_n(\vartheta)|\leq\frac{1}{2\left|\left|\sin\frac{\vartheta}{2}\right|\right|}$, hence the graph of the function $D_n(\vartheta)$ is “contained between”the graphs of $\frac{1}{2\sin\frac{\vartheta}{2}}$ and $-\frac{1}{2\sin\frac{\vartheta}{2}}$. Moreover, as n grows, over any interval of $\mathbb{R}$ the number of oscillations increases.

Exercise 1.0.17 *Define the sequence $\alpha\in\mathbb{R}^\infty$ by $\alpha_n:=\frac{1}{\log(n+1)}$. Prove that*

(i) $\alpha\notin\bigcup_{1\leq p<\infty}$.
(ii)$\alpha\in c_0\subset l_\infty$.

Exercise 1.0.18 *Define a sequence* $\alpha \in \mathbb{R}^\infty$ *by* $\forall n \in \mathbb{N} \cup \{0\}$ $a_n := 0$, $\forall n \in \mathbb{N} \setminus \{1\}$ $b_n := \frac{1}{\log n}$, $b_1 := 0$. *Then consider the corresponding sequence of functions on* $[0,2\pi]$ $\{s_n\}_n \in \mathbb{N}$ *where* $s_n(x) := \sum_{k=2}^{n} \frac{\sin(kx)}{\log(k)}$. *Prove that* $\forall x \in [0,2\pi]$ $\exists s_x \in \mathbb{R}$ *such that* $s_n(x) \longrightarrow s_x$.

With the notation of the previous exercise, s_x is denoted with $\sum_{k=2}^{\infty} \frac{\sin(kx)}{\log(k)}$, and then define the function $s : [0,2\pi] \longrightarrow \mathbb{R}$ as $s(x) := \sum_{k=2}^{\infty} \frac{\sin(kx)}{\log(k)}$.
One can prove that $s \notin L^1(0,2\pi)$, hence for s one does not compute the Fourier coefficients. For this reason $\sum_{k=2}^{\infty} \frac{\sin(kx)}{\log(k)}$ is not called Fourier series.

Exercise 1.0.19 *Define a sequence* $\widetilde{\alpha} \in \mathbb{R}^\infty$ *by* $\forall n \in \mathbb{N}$ $\widetilde{b}_n := 0$, $\forall n \in \mathbb{N} \setminus \{1\}$ $\widetilde{a}_n := \frac{1}{\log n}$, $\widetilde{a}_1 := 0$, $\widetilde{a}_0 := 0$. *Then consider the corresponding sequence of functions on* $[0,2\pi]$ $\{\widetilde{s}_n\}_n \in \mathbb{N}$ *where* $\widetilde{s}_n(x) := \sum_{k=2}^{n} \frac{\cos(kx)}{\log(k)}$. *Prove that* $\forall x \in [0,2\pi]$ $\exists \widetilde{s}_x \in \mathbb{R}$ *such that* $\widetilde{s}_n(x) \longrightarrow \widetilde{s}_x$.

Also this time, $\sum_{k=2}^{\infty} \frac{\cos(kx)}{\log(k)} := \widetilde{s}_x$, but the function $\widetilde{s} : [0,2\pi] \longrightarrow \mathbb{R}$ defined as $\widetilde{s}(x) := \sum_{k=2}^{\infty} \frac{\cos(kx)}{\log(k)}$ is in $L^1(0,2\pi)$, hence one can compute its Fourier coefficients. $\sum_{k=2}^{\infty} \frac{\cos(kx)}{\log(k)}$ is therefore a Fourier series. Judging by looks, this is surprising, given the similarity of $\sum_{k=2}^{\infty} \frac{\cos(kx)}{\log(k)}$ and $\sum_{k=2}^{\infty} \frac{\sin(kx)}{\log(k)}$.

Define $\mathbb{T} := [0,2\pi]$ and

$$A(\mathbb{T}) := \{s \in C([0,2\pi]) : \exists \alpha \in l_1 \text{ s.t. } s_n(x) \longrightarrow s(x)\} \tag{1.58}$$

Theorem 1.0.7 (Fatou, 1906) $s \in A(\mathbb{T}) \quad \Leftrightarrow \quad \forall x \in [0,2\pi] \quad s_n(x)$ *is absolutely convergent, i.e.* $\frac{a_0}{2} + \sum_{k=1}^{\infty} |a_k \cos(kx) + b_k \sin(kx)| < +\infty$.

$A(\mathbb{T})$ is a subset of $C([0,2\pi])$ isomorphic to l_1. If $f \in C([0,2\pi]) \setminus A(\mathbb{T})$, then the sequence α of Fourier coefficients of f surely belongs to l_2 since $C([0,2\pi]) \subset L^2(0,2\pi)$, but, more than this, it can be proved that $\forall q \in \mathbb{R}$, $q > 1$, $\alpha \in l_q$ ($\alpha \notin l_1$).
Let $0 < \beta \leq 1$. Define the set of Hölder continuous functions with exponent β as

$$C^{0,\beta}(\mathbb{T}) = Lip_\beta(\mathbb{T}) := \{f \in \mathbb{R}^\mathbb{T} : \exists C \in \mathbb{R} \text{ s.t. } \forall x,y \in \mathbb{T} |f(y) - f(x)| \leq C|y-x|^\beta\} \tag{1.59}$$

(If $\beta = 1$ it is simply the space of Lipschitz continuous functions.) $C^{0,\beta}(\mathbb{T}) \subset C(\mathbb{T})$.
It can be proved that if $\beta > \frac{1}{2}$, then $C^{0,\beta}(\mathbb{T}) \subset A(\mathbb{T})$. Nevertheless, for $0 < \beta \leq 1$ $C^{0,\beta}(\mathbb{T}) \cap BV(\mathbb{T}) \subset A(\mathbb{T})$ holds ($BV(\mathbb{T})$ denoting the set of functions of bounded variation).

Lemma 1.0.2 (Riemann-Lebesgue) *If* $f \in L^1(0,2\pi)$ *then the sequence* α *of Fourier coefficients of* f *belongs to* c_0.

Proof. `-First step. Exists a subset of` $L^1(0,2\pi)$ `which is dense in` $L^1(0,2\pi)$ `(with the topology induced by the norm` $\|\cdot\|_k L^1$`) such that the thesis holds whenever` f `belongs to this subset.`

Notice that measurable simple functions over $[0,2\pi]$ are in $L^2(0,2\pi)$. A classical result in measure theory is that any measurable function can be approximated by measurable simple functions. Hence the set Q of all the measurable simple functions over $[0,2\pi]$ is in $L^1([0,2\pi])$ and is dense in $L^1([0,2\pi])$. Since $L^2(0,2\pi)\supset Q$, $L^2([0,2\pi])$ is dense in $L^1(0,2\pi)$.
If $g\in L^2(0,2\pi)$ then the sequence $\alpha(g)$ of Fourier coefficients of g belongs to l_2 (Theorem 1.0.6), therefore it belongs to c_0 (Exercise 1.0.8).
`-Second step. Proof of the theorem using the first step.`
Let $f\in L^1(0,2\pi)$. Since $L^2(0,2\pi)$ is dense in $L^1(0,2\pi)$, consider a sequence $\{g_n\}_n\in\mathbb{N}$ of functions in $L^2(0,2\pi)$ such that $g_n\xrightarrow{L^1}f$. Thanks to Exercise 1.0.12, $\forall m\in\mathbb{N}$

$$\alpha_m(f)=\alpha_m(f-g_n+g_n)=\alpha_m(f-g_n)+\alpha_m(g_n)$$

From the definition of Fourier coefficients it follows immediately that for $h\in L^1(0,2\pi)$ and $\forall k\in\mathbb{N}\quad|\alpha_k(h)|\le\frac{1}{\pi}\|h\|_{L^1}$. Therefore

$$|\alpha_m(f)|\le|\alpha_m(f-g_n)|+|\alpha_m(g_n)|\le\frac{1}{\pi}\|f-g_n\|_{L^1}+|\alpha_m(g_n)|$$

Fix $\varepsilon>0$. Since $g_n\xrightarrow{L^1}f$ exists $N_\varepsilon\in\mathbb{N}$ such that $\forall N\ge N_\varepsilon$ $\|f-g_N\|_{L^1}<\frac{\pi\varepsilon}{2}$. Since the thesis holds for functions in $L^2(0,2\pi)$, it holds for g_{N_ε}, therefore $\exists M_\varepsilon$ such that $\forall M>M_\varepsilon$ $|\alpha_M(g_{N_\varepsilon})|<\frac{\varepsilon}{2}$. Thus

$$\forall\varepsilon>0\quad\exists N_\varepsilon,M_\varepsilon \text{ s.t. } \forall M>M_\varepsilon\qquad|\alpha_M(f)|\le\frac{1}{\pi}\|f-g_{N_\varepsilon}\|_{L^1}+|\alpha_M(g_{N_\varepsilon})|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Therefore $\alpha_m(f)\xrightarrow{m\to\infty}0$, i.e. $\alpha(f)\in c_0$.

Example 1.0.3

Let $f\in C([0,2\pi])$ such that $f(x)=\gamma+\int_0^x g(y)\mathrm{d}y$ where $\gamma\in\mathbb{R}$, $g\in L^2(0,2\pi)$ with the property $\int_0^{2\pi}g(y)\mathrm{d}y=0$. Notice that $f(0)=f(2\pi)=\gamma$. Then $s_n\longrightarrow f$ uniformly.
The sequence α of Fourier coefficients of f is in l_1. A result of general theory is that $f'(x)=g(x)$ a.e. $x\in[0,2\pi]$. Then, as in the second step of the proof of Theorem 1.0.5, one can show that for $k\in\mathbb{N}$ (only the tail of α is significant)

$$a_k(f)=-\frac{1}{k}b_k(g)\quad\text{ and }\quad b_k(f)=\frac{1}{k}a_k(g)\tag{1.60}$$

Thus

$$\begin{aligned}\sum_{k=1}^{\infty}\Big[|a_k(f)|+|b_k(f)|\Big]&=\sum_{k=1}^{\infty}\left[\frac{1}{k}\right]\Big[|a_k(g)|+|b_k(g)|\Big]\\&\le\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\Big[|a_k(g)|+|b_k(g)|\Big]^2\right)^{\frac{1}{2}}<+\infty\end{aligned}$$

Therefore, using Theorem 1.0.1, $\exists s \in C([0,2\pi])$ such that $s_n \longrightarrow s$ uniformly. Conjecture 1.0.1 holds for f, hence $s_n \longrightarrow f$ a.e., hence $f = s$ a.e., but f, s are both continuous, so it has to be $f = s$, and the uniform convergence holds.

Example 1.0.4

Consider, for $\beta \in \mathbb{R} \smallsetminus \mathbb{Z}$, the function over $\mathbb{R}$ $\quad f(x) = \cos(\beta(x-\pi))$. Then $f'(x) = -\beta \sin(\beta(x-\pi))$, $f(x) = \cos(\beta\pi) - \beta \int_0^x \sin(\beta(t-\pi))\mathrm{d}t$ and $-\beta \int_0^{2\pi} \sin(\beta(t-\pi))\mathrm{d}t = \cos(\beta(2\pi-\pi)) - \cos(\beta(-\pi)) = 0$. Hence f satisfies the hypotheses set out in the example above. The result about convergence follows and thus, pointwise,

$$\cos(\beta(x-\pi)) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big[a_k \cos(kx) + b_k \sin(kx)\Big] \tag{1.61}$$

where, for $k \in \mathbb{N}$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} \cos(\beta(x-\pi))dx = \frac{1}{\pi}\left[\frac{\sin(\beta(x-\pi))}{\beta}\right]_0^{2\pi} = \frac{2\sin(\beta\pi)}{\pi\beta} \\
a_k &= \frac{1}{\pi}\int_0^{2\pi} \cos(\beta(x-\pi))\cos(kx)\mathrm{d}x \overset{y \overset{s}{=} x-\pi}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\beta y)\cos(k(y+\pi))\mathrm{d}y \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\beta y)\Big(\cos(ky)\overbrace{\cos(k\pi)}^{(-1)^k} - \sin(ky)\overbrace{\sin(k\pi)}^{0}\Big)\mathrm{d}y \\
&= \frac{(-1)^k}{\pi}\int_{-\pi}^{\pi} \cos(\beta y)\cos(ky)\mathrm{d}y \overset{Lemma\ 1.0.1}{=} \frac{(-1)^k}{\pi}\int_{-\pi}^{\pi} \frac{\cos((k+\beta)y) + \cos((k-\beta)y)}{2}\mathrm{d}y \\
&= \frac{(-1)^k}{2\pi}\left[\frac{\sin((k+\beta)y)}{k+\beta} + \frac{\sin((k-\beta)y)}{k-\beta}\right]_{-\pi}^{\pi} = \frac{(-1)^k}{2\pi}\left(2\frac{\sin((k+\beta)\pi)}{k+\beta} + 2\frac{\sin((k-\beta)\pi)}{k-\beta}\right) \\
&= \frac{k(-1)^k}{\pi(k^2-\beta^2)}\left(\sin\left(\frac{k+\beta}{\beta}\beta\pi\right) + \sin\left(\frac{k-\beta}{\beta}\beta\pi\right)\right) - \frac{\beta(-1)^k}{\pi(k^2-\beta^2)}\left(\sin\left(\frac{k+\beta}{\beta}\beta\pi\right) - \sin\left(\frac{k-\beta}{\beta}\beta\pi\right)\right) \\
&\overset{Lemma\ 1.0.1}{=} \frac{k(-1)^k}{\pi(k^2-\beta^2)}\overbrace{2\sin\left(\frac{k}{\beta}\beta\pi\right)}^{0}\cos(\beta\pi) - \frac{\beta(-1)^k}{\pi(k^2-\beta^2)}\overbrace{2\cos\left(\frac{k}{\beta}\beta\pi\right)}^{(-1)^k}\sin(\beta\pi) \\
&= \frac{2\beta}{\pi(\beta^2-k^2)}\sin(\beta\pi) \\
b_k &= \frac{1}{\pi}\int_0^{2\pi} \cos(\beta(x-\pi))\sin(kx)\mathrm{d}x \overset{y \overset{s}{=} x-\pi}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\beta y)\sin(k(y+\pi))\mathrm{d}y \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\beta y)\Big(\cos(ky)\overbrace{\sin(k\pi)}^{0} + \sin(ky)\overbrace{\cos(k\pi)}^{(-1)^k}\Big)\mathrm{d}y \\
&= \frac{(-1)^k}{\pi}\int_{-\pi}^{\pi} \cos(\beta y)\sin(ky)\mathrm{d}y \overset{\sin\vartheta = -\sin(-\vartheta)}{=} 0
\end{aligned}$$

Thus

$$\cos(\beta(x-\pi)) = \frac{2}{\pi}\sin(\beta\pi)\left\{\frac{1}{2\beta} + \beta\sum_{k=1}^{\infty}\frac{\cos(kx)}{\beta^2-k^2}\right\} \tag{1.62}$$

Recall that, for $\beta, x \in \mathbb{R}$,

$$\cosh(\beta x) = \frac{e^{\beta x}+e^{-\beta x}}{2}, \qquad \sinh(\beta x) = \frac{e^{\beta x}-e^{-\beta x}}{2} \tag{1.63}$$

Moreover, for $\beta, x \in \mathbb{R} \setminus \{0\}$,

$$\coth(\beta x) = \frac{e^{\beta x}+e^{-\beta x}}{e^{\beta x}-e^{-\beta x}} = \frac{\cosh(\beta x)}{\sinh(\beta x)} \tag{1.64}$$

Exercise 1.0.20 *Compute, for $k \in \mathbb{N}$ and $\beta \in \mathbb{R}$, $\int_{-\pi}^{\pi} e^{\beta x}\cos(kx)\mathrm{d}x$ and $\int_{-\pi}^{\pi} e^{\beta x}\sin(kx)\mathrm{d}x$.*

Solution of Exercise 1.0.20 Notice that $e^{\beta x+ikx} = e^{\beta}\cos(kx) + ie^{\beta}\sin(kx)$. Easily

$$D := \int_{-\pi}^{\pi} e^{\beta x+ikx}\,\mathrm{d}x = \left[\frac{e^{\beta x+ikx}}{\beta+ik}\right]_{-\pi}^{\pi} = \frac{e^{\beta\pi}}{\beta+ik}\cos(k\pi) - \frac{e^{-\beta\pi}}{\beta+ik}\cos(k\pi)$$
$$= \frac{2\sinh(\beta\pi)}{\beta+ik}(-1)^k \cdot \frac{\beta-ik}{\beta-ik} = \frac{2\beta\sinh(\beta\pi)(-1)^k}{\beta^2+k^2} - i\frac{2k\sinh(\beta\pi)(-1)^k}{\beta^2+k^2}$$

Hence

$$\int_{-\pi}^{\pi} e^{\beta}\cos(kx)\mathrm{d}x = \mathfrak{Re}(D) = \frac{2\beta\sinh(\beta\pi)(-1)^k}{\beta^2+k^2}$$
$$\int_{-\pi}^{\pi} e^{\beta}\sin(kx)\mathrm{d}x = \mathfrak{Im}(D) = -\frac{2k\sinh(\beta\pi)(-1)^k}{\beta^2+k^2}$$

Example 1.0.5

Define, for $\beta \in \mathbb{R} \setminus \mathbb{Z}$, the function over $\mathbb{R}$ $\quad f(x) = \cosh(\beta(x-\pi))$. Then $f'(x) = \beta\sinh(\beta(x-\pi))$, $f(x) = \cosh(\beta\pi) + \beta\int_0^x \sinh(\beta(t-\pi))\mathrm{d}t$ and $\beta\int_0^{2\pi}\sinh(\beta(t-\pi))\mathrm{d}t = \cosh(\beta(2\pi-\pi)) - \cosh(\beta(-\pi)) = 0$. Hence, again, f satisfies the hypotheses for the pointwise convergence of Fourier series.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi}\cosh(\beta(x-\pi))\mathrm{d}x = \frac{1}{\pi}\left[\frac{\sinh(\beta(x-\pi))}{\beta}\right]_0^{2\pi} = \frac{2\sinh(\beta\pi)}{\beta}$$

$$a_k = \frac{1}{\pi}\int_0^{2\pi}\cosh(\beta(x-\pi))\cos(kx)\mathrm{d}x \overset{y\overset{s}{=}x-\pi}{=} \frac{1}{\pi}\int_{-\pi}^{\pi}\cosh(\beta y)\cos(k(y+\pi))\mathrm{d}y$$

$$= \frac{(-1)^k}{\pi}\int_{-\pi}^{\pi}\cosh(\beta y)\cos(ky)\mathrm{d}y = \frac{(-1)^k}{2\pi}\int_{-\pi}^{\pi}(e^{\beta y} - e^{(-\beta)y})\cos(ky)\mathrm{d}y$$

$$\overset{Exercise\ 1.0.20}{=} \frac{(-1)^k}{2\pi}\left(\frac{2\beta\sinh(\beta\pi)(-1)^k}{\beta^2+k^2} - \frac{2\beta\sinh(-\beta\pi)(-1)^k}{\beta^2+k^2}\right) = \frac{2\beta\sinh(\beta\pi)}{\pi(\beta^2+k^2)}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi}\cosh(\beta(x-\pi))\sin(kx)\mathrm{d}x \overset{y\overset{s}{=}x-\pi}{=} \frac{1}{\pi}\int_{-\pi}^{\pi}\cosh(\beta y)\sin(k(y+\pi))\mathrm{d}y$$

$$= \frac{(-1)^k}{\pi}\int_{-\pi}^{\pi}\cosh(\beta y)\sin(ky)\mathrm{d}y \overset{\sin\vartheta=-\sin(-\vartheta)}{=} 0$$

Thus, pointwise,

$$\cosh(\beta(x-\pi)) = \frac{2}{\pi}\sinh(\beta\pi)\left\{\frac{1}{2\beta} + \beta\sum_{k=1}^{\infty}\frac{\cos(kx)}{\beta^2+k^2}\right\} \tag{1.65}$$

In particular, if $x = 0$

$$\cosh(\beta\pi) = \frac{2}{\pi}\sinh(\beta\pi)\left\{\frac{1}{2\beta} + \beta\sum_{k=1}^{\infty}\frac{1}{\beta^2+k^2}\right\} \tag{1.66}$$

And therefore, by means of Fourier series, one gets

$$\sum_{k=1}^{\infty}\frac{1}{\beta^2+k^2} = \frac{\pi\beta\coth\pi\beta - 1}{2\beta^2} \qquad \text{for } \beta\notin\mathbb{Z} \tag{1.67}$$

Considering the limit for β approaching an integer different from 0, one also gets that the previous formula holds for $\beta\in\mathbb{R}\setminus\{0\}$. The case $\beta = 0$ is dealt with in the following exercise.

Exercise 1.0.21 *Using 1.67, show again that* $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.

Solution of Exercise 1.0.21. Define for $x\in\mathbb{R}$, $x\neq 0$,

$$D(x) := \frac{x\coth x - 1}{2x^2} = \frac{x\frac{e^{2x}+1}{e^{2x}-1} - 1}{2x^2} = \frac{x(e^{2x}+1) - e^{2x} + 1}{2x^2(e^{2x}-1)}$$

Since

$$x(e^{2x}+1) - e^{2x} + 1 \sim_0 x(2 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6}) - 1 - 2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{6} + 1 \sim_0 \frac{2}{3}x^3$$

$$2x^2(e^{2x}-1) \sim_0 2x^2(2x) = 4x^3$$

it follows that

$$\frac{\pi^2}{6} = \lim_{x\to 0} \pi^2 \frac{\frac{2}{3}x^3}{4x^3} = \lim_{x\to 0} \pi^2 D(x) = \lim_{\beta\to 0} \frac{\pi\beta\coth\pi\beta - 1}{2\beta^2} = \lim_{\beta\to 0} \sum_{k=1}^{\infty} \frac{1}{\beta^2+k^2} \overset{\dagger}{=} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

Indeed, to prove †, (recall that $\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2$ was shown at the beginning of these notes)

$$0 \leq \sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2+\beta^2} = \beta^2 \sum_{k=1}^{n} \frac{1}{k^2(k^2+\beta^2)} < \beta^2 \sum_{k=1}^{n} \frac{1}{k^2} \leq 2\beta^2$$

And so it easily follows that

$$\lim_{\beta\to 0} \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2+\beta^2} \right) = 0$$

Hence, from basic limit properties,

$$\begin{aligned}
\lim_{\beta\to 0} \sum_{k=1}^{\infty} \frac{1}{\beta^2+k^2} &= \lim_{\beta\to 0} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\beta^2+k^2} \\
&= \lim_{\beta\to 0} \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k^2} - \left(\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2+\beta^2} \right) \right) \\
&= \lim_{\beta\to 0} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k^2} - \lim_{\beta\to 0} \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2+\beta^2} \right) = \sum_{k=1}^{\infty} \frac{1}{k^2}
\end{aligned}$$

Now consider, for $\beta \in \mathbb{R} \setminus \mathbb{Z}$, the equality 1.62 when $x = 0$. That leads to † in the following computation (in which some details about convergence are skipped):

$$\begin{aligned}
&\frac{d}{d\beta} \log \frac{(-1)^{\lfloor|\beta|\rfloor} \sin(\pi\beta)}{\beta} = \frac{d}{d\beta} \log \left| \frac{\sin(\pi\beta)}{\beta} \right| = \frac{d}{d\beta} \left[\log|\sin(\pi\beta)| - \log|\beta| \right] \\
&= \frac{\pi\cos(\pi\beta)}{\sin(\pi\beta)} - \frac{1}{\beta} \overset{\dagger}{=} \sum_{k=1}^{\infty} \frac{2\beta}{\beta^2-k^2} = \sum_{k=1}^{\lfloor|\beta|\rfloor} \frac{2\frac{\beta}{k^2}}{\frac{k^2}{\beta^2}-1} + \sum_{k=\lfloor|\beta|\rfloor+1}^{\infty} \frac{-2\frac{\beta}{k^2}}{1-\frac{k^2}{\beta^2}} \\
&= \sum_{k=1}^{\lfloor|\beta|\rfloor} \frac{d}{d\beta} \log\left(\frac{k^2}{\beta^2}-1\right) + \sum_{k=\lfloor|\beta|\rfloor+1}^{\infty} \frac{d}{d\beta} \log\left(1-\frac{k^2}{\beta^2}\right) = \frac{d}{d\beta} \sum_{k=1}^{\lfloor|\beta|\rfloor} \log\left(\frac{k^2}{\beta^2}-1\right) + \frac{d}{d\beta} \sum_{k=\lfloor|\beta|\rfloor+1}^{\infty} \log\left(1-\frac{k^2}{\beta^2}\right) \\
&= \frac{d}{d\beta} \log \prod_{k=1}^{\lfloor|\beta|\rfloor} \left(\frac{k^2}{\beta^2}-1\right) + \frac{d}{d\beta} \log \prod_{k=\lfloor|\beta|\rfloor+1}^{\infty} \left(1-\frac{k^2}{\beta^2}\right) \\
&= \frac{d}{d\beta} \log(-1)^{\lfloor|\beta|\rfloor} \prod_{k=1}^{\lfloor|\beta|\rfloor} \left(1-\frac{k^2}{\beta^2}\right) + \frac{d}{d\beta} \log \prod_{k=\lfloor|\beta|\rfloor+1}^{\infty} \left(1-\frac{k^2}{\beta^2}\right) = \frac{d}{d\beta} \log(-1)^{\lfloor|\beta|\rfloor} \prod_{k=1}^{\infty} \left(1-\frac{k^2}{\beta^2}\right)
\end{aligned}$$

Therefore $\exists c \in \mathbb{R}$ such that for $\beta \in (0,1)$

$$\log \frac{(-1)^{\lfloor|\beta|\rfloor} \sin(\pi\beta)}{\beta} = \log(-1)^{\lfloor|\beta|\rfloor} \prod_{k=1}^{\infty} \Big(1 - \frac{k^2}{\beta^2}\Big) + c \tag{1.68}$$

hence

$$\frac{(-1)^{\lfloor|\beta|\rfloor} \sin(\pi\beta)}{\beta} = e^c (-1)^{\lfloor|\beta|\rfloor} \prod_{k=1}^{\infty} \Big(1 - \frac{k^2}{\beta^2}\Big) \tag{1.69}$$

i.e.

$$\frac{\sin(\pi\beta)}{\beta} = e^c \prod_{k=1}^{\infty} \Big(1 - \frac{k^2}{\beta^2}\Big) \tag{1.70}$$

Taking the limit for $\beta \to 0^+$ it can be seen that $e^c = \pi$. Actually

$$\frac{\sin(\pi\beta)}{\pi\beta} = \prod_{k=1}^{\infty} \Big(1 - \frac{k^2}{\beta^2}\Big) \qquad \forall \beta \in \mathbb{R} \setminus \{0\} \tag{1.71}$$

This result was known in the eighteenth century and Euler used it as a tool for proving $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

1.1 Cesàro convergence

Let $\{a_n\}_n$ be a real-valued sequence. Then, if $a \in \mathbb{R}$, the sequence converges to a in the Cesàro sense, written

$$a_n \xrightarrow{(C)} a \tag{1.72}$$

if

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \longrightarrow a \tag{1.73}$$

i.e. if

$$\forall \varepsilon > 0 \qquad \exists n_\varepsilon \in \mathbb{N} \quad : \quad \forall n \geq n_\varepsilon \qquad \Big| \frac{a_1 + a_2 + \cdots + a_n}{n} - a \Big| < \varepsilon \tag{1.74}$$

Proposition 1.1.1 *Let $\{a_n\}_n$ be a real-valued sequence, $a \in \mathbb{R}$. Then*

$$a_n \longrightarrow a \qquad \Rightarrow \qquad a_n \xrightarrow{(C)} a \tag{1.75}$$

Proof. Fix $\varepsilon > 0$, then, thanks to the classical definition of convergence

$$\exists n_\varepsilon \in \mathbb{N} \quad : \quad \forall n \geq n_\varepsilon \quad |a_n - a| < \varepsilon \tag{1.76}$$

Suppose $n \geq n_\varepsilon$.

$$\left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right| = \left|\frac{(a_1-a)+(a_2-a)+\cdots+(a_n-a)}{n}\right|$$
$$\leq \frac{|a_1-a|+|a_2-a|+\cdots+|a_n-a|}{n}$$
$$= \frac{|a_1-a|+|a_2-a|+\cdots+|a_{n_\varepsilon-1}-a|}{n} + \frac{|a_{n_\varepsilon}-a|+|a_{n_\varepsilon+1}-a|+\cdots+|a_n-a|}{n}$$
$$\leq \frac{|a_1-a|+|a_2-a|+\cdots+|a_{n_\varepsilon-1}-a|}{n} + \frac{n-n_\varepsilon+1}{n}\varepsilon$$

In order to prove the existence of $\lim_{n\to\infty}\left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right|$, consider

$$0 \leq \limsup_n \left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right|$$
$$\leq \limsup_n \frac{|a_1-a|+|a_2-a|+\cdots+|a_{n_\varepsilon-1}-a|}{n} + \limsup_n \frac{n-n_\varepsilon+1}{n}\varepsilon = \varepsilon$$

Thus, since inequalities hold for all $\varepsilon > 0$, it has to be

$$\limsup_n \left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right| = 0$$

Hence

$$0 \leq \liminf_n \left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right| \leq \limsup_n \left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right| \leq 0$$

yields both the existence of the limit and its value:

$$\lim_{n\to\infty} \left|\frac{a_1+a_2+\cdots+a_n}{n}-a\right| = 0$$

Observation 1.1.1 *If a real-valued sequence converges according to Cesàro, it does not necessarily converge in the classical sense, i.e. the reverse implication in Proposition 1.1.1 does not hold.*

For instance, consider the real-valued sequence $\{a_n\}_{n\in\mathbb{N}}$ that for $n \in \mathbb{N}$ is defined by

$$a_n := \begin{cases} -1 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \tag{1.77}$$

Then

$$\limsup_n a_n = 1$$
$$\liminf_n a_n = -1$$

and thus $\nexists \lim_n a_n$, so $\{a_n\}_n$ does not converge in the classical sense.
Nevertheless, defining

$$\alpha_n := \frac{\sum_{i=1}^{n} a_i}{n} \tag{1.78}$$

it is easy to check that

$$\alpha_n := \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases} \tag{1.79}$$

Hence the sequence $\{\alpha_n\}_n$ converges to 0, and this means exactly that

$$a_n \xrightarrow{(C)} 0 \tag{1.80}$$

The notion of convergence in the Cesàro sense can be extended to general normed spaces.

Definition 1.1.1 *Let $(X, \|\cdot\|)$ be a normed space. Let $\{a_n\}_n$ be a sequence in X, $a \in X$. Then $\{a_n\}_n$ converges to a in the Cesàro sense, written*

$$a_n \xrightarrow{(C)} a \tag{1.81}$$

if

$$\forall \varepsilon > 0 \qquad \exists n_\varepsilon \in \mathbb{N} \quad : \quad \forall n \geq n_\varepsilon \qquad \left\| \frac{a_1 + a_2 + \cdots + a_n}{n} - a \right\| < \varepsilon \tag{1.82}$$

Moreover

Proposition 1.1.2 *Let $(X, \|\cdot\|)$ be a normed space. Let $\{a_n\}_n$ be a sequence in X, $a \in X$. Then*

$$a_n \longrightarrow a \qquad \Rightarrow \qquad a_n \xrightarrow{(C)} a \tag{1.83}$$

Indeed, *mutatis mutandis*, the same proof of Proposition 1.1.1 holds: essentially, that proof relied upon the triangular inequality, and the triangular inequality holds in any normed space.

Let $\alpha \in \mathbb{R}^\infty$. Construct the corrisponding sequence of functions $\{s_n\}_{n \in \mathbb{N} \cup \{0\}}$. Instead of questioning about the pointwise convergence of the sequence, one can wonder whether for $x \in [0, 2\pi]$ the sequence $\{s_n(x)\}_{n \in \mathbb{N} \cup \{0\}}$ converges in the Cesàro sense: define the sequence $\{\sigma_n\}_{n \in \mathbb{N} \cup \{0\}}$ of functions by

$$\sigma_n(x) := \frac{\sum_{i=0}^{n} s_i(x)}{n+1} \tag{1.84}$$

Notice that

$$\sigma_n(x) = \frac{\sum_{i=0}^{n} \left(\frac{a_0}{2} + \sum_{k=1}^{i} [a_k \cos(kx) + b_k \sin(kx)] \right)}{n+1} \tag{1.85}$$

$$= \frac{a_0}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) [a_k \cos(kx) + b_k \sin(kx)] \tag{1.86}$$

that is, σ_n and s_n have similar forms; σ_n has coefficients that improve convergence. Indeed

Remark 2 *If there exists a function over $[0,2\pi]$ such that for $x \in [0,2\pi]$ $s_n(x) \longrightarrow s(x)$, then, thanks to Proposition 1.1.1, $\sigma_n(x) \longrightarrow s(x)$.*

As a particular case, there's the one concerning the sequence α of Fourier coefficients of a function $f \in L^1(0,2\pi)$. It was shown in 1.57 that $s_n(x) = \frac{1}{\pi}(f * D_n)(x)$. Define, for $x \in [0,2\pi]$, the Fejér kernel as

$$K_n(x) := \frac{\sum_{i=0}^n D_i(x)}{n+1} \tag{1.87}$$

Hence, by the linearity of the integral,

$$\begin{aligned}\sigma_n(x) &= \frac{\sum_{i=0}^n \left(\frac{1}{\pi}\int_0^{2\pi} f(t) D_n(x-t)\mathrm{d}t\right)}{n+1} = \frac{1}{\pi}\int_0^{2\pi} f(t)\frac{\sum_{i=0}^n D_n(x-t)}{n+1}\mathrm{d}t \\ &= \frac{1}{\pi}\int_0^{2\pi} f(t) K_n(x-t)\mathrm{d}t = \frac{1}{\pi}(f * K_n)(x)\end{aligned}$$

The Fejér kernels, unlike Dirichlet kernels, enjoy the property of being always non-negative:

Exercise 1.1.1 *Prove*

$$K_n(x) = \frac{1}{2(n+1)}\left(\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\right)^2 \tag{1.88}$$

Solution of Exercise 1.1.1.

$$\begin{aligned}K_n(x) &= \frac{\sum_{k=0}^n D_k(x)}{n+1} \overset{1.54}{=} \frac{\sum_{k=0}^n \sin\frac{(2k+1)x}{2}}{2(n+1)\sin\frac{x}{2}} = \frac{\sum_{k=0}^n \left(\sin(kx)\cos\frac{x}{2} + \cos(kx)\sin\frac{x}{2}\right)}{2(n+1)\sin\frac{x}{2}} \\ &= \frac{\cos\frac{x}{2}}{2(n+1)\sin\frac{x}{2}}\sum_{k=1}^n \sin(kx) + \frac{\sin\frac{x}{2}}{2(n+1)\sin\frac{x}{2}}\sum_{k=0}^n \cos(kx) \\ &\overset{1.56 \text{ and } 1.55}{=} \frac{\cos\frac{x}{2}}{2(n+1)\sin\frac{x}{2}}\,\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\,\sin\frac{nx}{2} + \frac{\sin\frac{x}{2}}{2(n+1)\sin\frac{x}{2}}\,\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\,\cos\frac{nx}{2} \\ &= \frac{\sin\frac{(n+1)x}{2}}{2(n+1)\sin\frac{x}{2}}\,\frac{\cos\frac{x}{2}\sin\frac{nx}{2} + \cos\frac{nx}{2}\sin\frac{x}{2}}{\sin\frac{x}{2}} = \frac{1}{2(n+1)}\left(\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}\right)^2\end{aligned}$$

Notice that $\forall n \in \mathbb{N}$

$$\int_0^{2\pi} D_n(t)\mathrm{d}t = \int_0^{2\pi}\Big(\frac{1}{2}+\sum_{k=1}^{n}\cos(kt)\Big)\mathrm{d}t = \int_0^{2\pi}\frac{1}{2}\mathrm{d}t + \sum_{k=1}^{n}\int_0^{2\pi}\cos(kt)\mathrm{d}t = \pi \tag{1.89}$$

$$\int_0^{2\pi} K_n(t)\mathrm{d}t = \int_0^{2\pi}\frac{\sum_{i=0}^{n} D_i(x)}{n+1}\mathrm{d}t = \frac{\sum_{i=0}^{n}\int_0^{2\pi} D_i(x)\mathrm{d}t}{n+1} = \pi \tag{1.90}$$

Moreover, for $n \to \infty$ D_n just increases the number of oscillations in an interval but it does not "lower"itself, whereas for $x \in \mathbb{R} \setminus \pi\mathbb{Z}$ $K_n(x) \to 0$.

Theorem 1.1.1 (Fejér) *If $f \in C([0,2\pi])$ and $f(0) = f(2\pi)$ then $\sigma_n \longrightarrow f$ uniformly.*

Proof. From 1.90 and observing that K_n is periodic, for $x \in [0,2\pi]$

$$f(x) = f(x)\frac{1}{\pi}\int_0^{2\pi} K_n(y)\mathrm{d}y = f(x)\frac{1}{\pi}\int_0^{2\pi} K_n(x-y)\mathrm{d}y \tag{1.91}$$

Fix $\varepsilon > 0$. Since $f \in C([0,2\pi])$ $\exists\delta_\varepsilon > 0$ such that, for $x,y \in [0,2\pi]$, if $|x-y| < \delta_\varepsilon$ then $|f(x)-f(y)| < \varepsilon$. Moreover, from a well-known Weierstrass theorem, $\exists M \in \mathbb{R}$ such that $M = \max_{[0,2\pi]} f$. Therefore, for $x \in [0,2\pi]$,

$$\begin{aligned}
|\sigma_n(x) - f(x)| &= \left|\frac{1}{\pi}\int_0^{2\pi} f(y)K_n(x-y)\mathrm{d}y - f(x)\right| \overset{1.91}{=} \left|\frac{1}{\pi}\int_0^{2\pi}[f(y)-f(x)]K_n(x-y)\mathrm{d}y\right| \\
&\le \frac{1}{\pi}\int_0^{2\pi}|f(y)-f(x)|K_n(x-y)\mathrm{d}y \\
&= \frac{1}{\pi}\int_{|y-x|<\delta_\varepsilon}|f(y)-f(x)|K_n(x-y)\mathrm{d}y + \frac{1}{\pi}\int_{|y-x|\ge\delta_\varepsilon}|f(y)-f(x)|K_n(x-y)\mathrm{d}y \\
&< \frac{\varepsilon}{\pi}\int_{|y-x|<\delta_\varepsilon}K_n(x-y)\mathrm{d}y + \frac{2M}{\pi}\int_{|y-x|\ge\delta_\varepsilon}K_n(x-y)\mathrm{d}y \\
&\le \varepsilon + \frac{2M}{\pi}\int_{|t|\ge\delta_\varepsilon}K_n(t)\mathrm{d}t \overset{\ddagger}{<} \varepsilon + \frac{C}{n}
\end{aligned}$$

where $C \in \mathbb{R}$. Indeed, ‡ holds since in a fixed interval having empty intersection with $\pi\mathbb{Z}$ K_n goes to zero with speed controlled by the coefficient $\frac{1}{n+1} \sim_\infty \frac{1}{n}$. Then

$$0 \le \liminf_n |\sigma_n(x) - f(x)| \le \limsup_n |\sigma_n(x) - f(x)| < \varepsilon$$

And this means that $\exists \lim_n |\sigma_n(x) - f(x)|$ and $\lim_n |\sigma_n(x) - f(x)| = 0$, i.e. the pointwise convergence $\sigma_n \to f$ holds. Since the evaluation in terms of $\varepsilon + \frac{C}{n}$ does not depend on the chosen x, the convergence is also uniform.

In particular, the sequence $\{s_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges to $f(x)$ in the Cesàro sense. However, it can be seen that the pointwise convergence in general does not hold. Notice that the above proof would fail if s_n substituted for σ_n: D_n, unlike K_n, does not converge to the zero function in any interval. The first example of $f \in C([0,2\pi])$ such that for some $x \in [0,2\pi]$ $s_n(x) \to +\infty$ is due to Du Bois Reymond, around about 1875.

Theorem 1.1.2 *If $f \in L^1(0,2\pi)$ then $\sigma_n \xrightarrow{L^1} f$.*

Proof. Extend f to a function on the whole real line (by loose language, the extension will be referred to as f): if $t \in \mathbb{R} \setminus 2\pi\mathbb{Z}$, there are a unique $j \in \mathbb{N}$ and a unique $x \in [0,2\pi]$ such that $t = x + 2j\pi$, thus define $f(t) := f(x)$. Since the Lebesge measure of $2\pi\mathbb{Z}$ is zero and $f \in L^1(0,2\pi)$, it is feasible to avoid defining the extension over $2\pi\mathbb{Z}$.
For $y \in \mathbb{R}$, define $f_y; [0,2\pi] \longrightarrow \mathbb{R}$ as $f_y(t) := f(t-y)$. A general result states that $\lim_{y\to 0} \|f_y - f\|_{L^1(0,2\pi)} = 0$.
Fix $\varepsilon > 0$. Therefore $\exists \delta_\varepsilon > 0$ such that if $|y| < \delta_\varepsilon$ then $\|f_y - f\|_{L^1(0,2\pi)} < \varepsilon$.

$$\begin{aligned}
\|\sigma_n - f\|_{L^1(0,2\pi)} &= \int_0^{2\pi} |\sigma_n(x) - f(x)| \mathrm{d}x \overset{1.91}{=} \frac{1}{\pi} \int_0^{2\pi} \left| \int_0^{2\pi} [f(y) - f(x)] K_n(x-y) \mathrm{d}y \right| \mathrm{d}x \\
&= \frac{1}{\pi} \int_0^{2\pi} \left| \int_0^{2\pi} [f(x-y) - f(x)] K_n(y) \mathrm{d}y \right| \mathrm{d}x \leq \frac{1}{\pi} \int_0^{2\pi} \left(\int_0^{2\pi} |f(x-y) - f(x)| K_n(y) \mathrm{d}y \right) \mathrm{d}x \\
&\overset{\text{Fubini}}{=} \frac{1}{\pi} \int_0^{2\pi} K_n(y) \left(\int_0^{2\pi} |f(x-y) - f(x)| \mathrm{d}x \right) \mathrm{d}y = \frac{1}{\pi} \int_0^{2\pi} K_n(y) \|f_y - f\|_{L^1(0,2\pi)} \mathrm{d}y \\
&= \frac{1}{\pi} \int_{|y|<\delta_\varepsilon} K_n(y) \|f_y - f\|_{L^1(0,2\pi)} \mathrm{d}y + \frac{1}{\pi} \int_{|y|\geq\delta_\varepsilon} K_n(y) \|f_y - f\|_{L^1(0,2\pi)} \mathrm{d}y \\
&< \varepsilon + \frac{2\|f\|_{L^1(0,2\pi)}}{\pi} \int_{|y|\geq\delta_\varepsilon} K_n(y) \mathrm{d}y \ \leq\ \varepsilon + \frac{C}{n}
\end{aligned}$$

where $C \in \mathbb{R}$ and the latter inequality holds for the reasons stated towards the end of the proof of Fejér theorem. Again,

$$0 \leq \liminf_n \|\sigma_n - f\|_{L^1(0,2\pi)} \leq \limsup_n \|\sigma_n - f\|_{L^1(0,2\pi)} < \varepsilon \tag{1.92}$$

thereby $\exists \lim_n \|\sigma_n - f\|_{L^1(0,2\pi)}$ and $\lim_n \|\sigma_n - f\|_{L^1(0,2\pi)} = 0$.

If $f \in L^1(0,2\pi)$, an already mentioned example by Kolmogorov (around about 1926) shows that in general the same conclusion for the sequence $\{s_n\}_n$ does not hold.
Theorem 1.1.2 triggers a new proof of Theorem 1.0.5: suppose the Fourier coefficients of a $f \in L^1(0,2\pi)$ are all zero. Then $\forall n \in \mathbb{N}$ $s_n \equiv 0 \equiv \sigma_n$, therefore $0 = \lim_n \|\sigma_n - f\|_{L^1(0,2\pi)} = \lim_n \|f\|_{L^1(0,2\pi)} = \|f\|_{L^1(0,2\pi)}$, i.e. $f = 0$ a.e..

Example 1.1.1

Consider the function over $[0,2\pi]$ $h(x) := \frac{\pi - x}{2}$ already considered just after Exercise 1.0.14. Extend h to the whole real line as was described beginning the proof of Theorem 1.1.2. Let $h(w^+)$ (resp. $h(w^-)$) denote the limit of h from the right (resp. the left). Then $\forall x \in [0,2\pi]$

$$s_n(x) \longrightarrow \frac{h(x^+)+h(x^-)}{2} \tag{1.93}$$

Moreover, for a real $0<\delta<\pi$, $s_n \longrightarrow h$ uniformly in $[\delta, 2\pi-\delta]$.

Proof. Define the function over $\mathbb{R}$ $\widetilde{h}(x) := (1-\cos x)h(x)$. $\widetilde{h}$ is continuous since for $x \notin 2\pi\mathbb{Z}$ it is the product of continuous functions, and for $w \in 2\pi\mathbb{Z}$ $\widetilde{h}(w)=0$ and also the limits are 0.
The derivative of $\widetilde{h}$ exists in every point of $\mathbb{R}$:

$$g(x) := \widetilde{h}'(x) = \begin{cases} \frac{2h(x)\sin(x)+\cos(x)-1}{2} & \text{if } x \notin 2\pi\mathbb{Z} \\ 0 & \text{if } x \in 2\pi\mathbb{Z} \end{cases} \tag{1.94}$$

Indeed, if $x = 2\pi m$ for $m \in \mathbb{Z}$,

$$\frac{\widetilde{h}'(2\pi m+r)-\widetilde{h}'(2\pi m)}{(2\pi m+r)-2\pi m} = \frac{(1-\cos(2\pi m+r))h(2\pi m+r)-(1-\cos 2\pi m)h(2\pi m)}{r}$$
$$= \frac{(1-\cos(r))h(2\pi m+r)}{r} \sim_0 \frac{\frac{r^2}{2}\frac{(1-2m)\pi}{2}}{r} = \frac{r(1-2m)\pi}{4} \xrightarrow{r\to 0} 0$$

g is bounded, hence $g \in L^2(0,2\pi)$. Therefore, for $c \in \mathbb{R}$

$$\widetilde{h}(x) = c + \int_0^{2\pi} g(\xi)\mathrm{d}\xi \tag{1.95}$$

$\widetilde{h}$ satisfies the hypotheses set out in Example 1.0.3, thus, if $\{\widetilde{s}_n\}_n$ is the sequence of functions associated to $\widetilde{h}$, it follows that $\widetilde{s}_n \longrightarrow \widetilde{h}$ uniformly, i.e. a stronger convergence than the one given by Fejér theorem holds. Using repeatedly Exercise 1.0.14, the Fourier coefficients of $\widetilde{h}$ are (a_k, b_k denote the Fourier coefficients of h):

$$\widetilde{a}_0 = \frac{1}{\pi}\int_0^{2\pi}(1-\cos x)\frac{\pi-x}{2}\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\mathrm{d}x - \frac{1}{\pi}\int_0^{2\pi}\cos x\frac{\pi-x}{2}\mathrm{d}x = a_0 - a_1 = 0$$
$$\widetilde{a}_k = \frac{1}{\pi}\int_0^{2\pi}(1-\cos x)\frac{\pi-x}{2}\cos(kx)\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\cos(kx)\mathrm{d}x - \frac{1}{\pi}\int_0^{2\pi}\cos x\frac{\pi-x}{2}\cos(kx)\mathrm{d}x$$
$$\overset{\text{Lemma 1.0.1}}{=} a_k - \frac{1}{2\pi}\int_0^{2\pi}\frac{\pi-x}{2}(\cos((k+1)x)+\cos((k-1)x))\mathrm{d}x = a_k - \frac{a_{k+1}-a_{k-1}}{2} = 0$$
$$\widetilde{b}_k = \frac{1}{\pi}\int_0^{2\pi}(1-\cos x)\frac{\pi-x}{2}\sin(kx)\mathrm{d}x = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\sin(kx)\mathrm{d}x - \frac{1}{\pi}\int_0^{2\pi}\cos x\frac{\pi-x}{2}\sin(kx)\mathrm{d}x$$
$$\overset{\text{Lemma 1.0.1}}{=} b_k - \frac{1}{2\pi}\int_0^{2\pi}\frac{\pi-x}{2}(\sin((k+1)x)+\sin((k-1)x))\mathrm{d}x$$
$$= \begin{cases} b_1 - \frac{b_2}{2} = 1 - \frac{\frac{1}{2}}{2} = \frac{3}{4} & \text{if} \quad k=1 \\ b_k - \frac{b_{k+1}+b_{k-1}}{2} = \frac{1}{k} - \frac{\frac{1}{k+1}+\frac{1}{k-1}}{2} = -\frac{1}{k(k^2-1)} & \text{if} \quad k \in \mathbb{N} \setminus \{1\} \end{cases}$$

Therefore

$$\widetilde{s}_n(x) = \frac{3}{4}\sin x + \sum_{k=2}^{n} \frac{\sin(kx)}{k(1-k^2)} = -\frac{1}{4}\sin x + \sin x + \sum_{k=2}^{n}\left(\frac{1}{k} - \frac{1}{2(k+1)} - \frac{1}{2(k-1)}\right)\sin(kx)$$

$$= \sin x + \sum_{k=2}^{n}\frac{\sin(kx)}{k} - \frac{1}{2}\left(\frac{1}{2}\sin x + \sum_{k=2}^{n}\frac{\sin(kx)}{k+1} + \sum_{k=2}^{n}\frac{\sin(kx)}{k-1}\right)$$

$$= s_n(x) - \frac{1}{2}\left(\frac{1}{2}\sin x + \sum_{k=2}^{n-2}\frac{\sin(kx)}{k+1} + \frac{\sin((n-1)x)}{n} + \frac{\sin(nx)}{n+1} + \sum_{k=4}^{n}\frac{\sin(kx)}{k-1} + \frac{\sin(2x)}{1} + \frac{\sin(3x)}{2}\right)$$

$$= s_n(x) - \frac{1}{2}\left(\sin(2x) + \frac{\sin(3x)+\sin x}{2} + \sum_{i=3}^{n-1}\frac{\sin((i+1)x)+\sin((i-1)x)}{i} + \frac{\sin((n-1)x)}{n}\right) - \frac{\sin(nx)}{2(n+1)}$$

$$\overset{\text{Lemma 1.0.1}}{=} s_n(x) - \frac{1}{2}\left(2\sin(x)\cos(x) + \sin(2x)\cos(x) + 2\sum_{i=3}^{n-1}\frac{\sin(ix)\cos(x)}{i} + \frac{\sin((n-1)x)}{n}\right) - \frac{\sin(nx)}{2(n+1)}$$

$$= s_n(x) - \frac{1}{2}\left(2\cos(x)\sum_{k=1}^{n}\frac{\sin(kx)}{k} - \frac{2\sin(nx)\cos(x)}{n} + \frac{\sin((n-1)x)}{n}\right) - \frac{\sin(nx)}{2(n+1)}$$

$$\overset{\text{Lemma 1.0.1}}{=} s_n(x) - \frac{1}{2}\left(2s_n(x)\cos(x) - \frac{\sin((n+1)x)+\sin((n-1)x)}{n} + \frac{\sin((n-1)x)}{n}\right) - \frac{\sin(nx)}{2(n+1)}$$

$$= (1-\cos(x))\,s_n(x) + \frac{\sin((n+1)x)}{2n} - \frac{\sin(nx)}{2(n+1)}$$

Thereby

$$(1-\cos(x))\,s_n(x) = \widetilde{s}_n(x) - \frac{\sin((n+1)x)}{2n} + \frac{\sin(nx)}{2(n+1)}$$

As it has been already observed, $\widetilde{s}_n \longrightarrow \widetilde{h}$ uniformely. Moreover for every x $|\sin((n+1)|, |\sin(nx)| \le 1$, hence also the convergences $-\frac{\sin((n+1)x)}{2n} \longrightarrow 0$ and $\frac{\sin(nx)}{2(n+1)} \longrightarrow 0$ are uniform. Thus it easily follows that

$$(1-\cos(x))\,s_n(x) \longrightarrow \widetilde{h}(x) = (1-\cos(x))\,h(x) \qquad \text{uniformly} \tag{1.96}$$

Hence, for $\varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ such that if $N > N_\varepsilon$ for $x \in [0,2\pi]$ $\quad |(1-\cos(x))| \cdot |s_N(x) - h(x)| = |(1-\cos(x))\,s_N(x) - (1-\cos(x))\,h(x)| < \varepsilon$ Fix $0 < \delta < \pi$. For $x \in [\delta, 2\pi-\delta]$ $0 < |1-\cos\delta| \le |(1-\cos(x))| < 1$. In particular $|(1-\cos(x))| \ne 0$ and $\frac{1}{|(1-\cos(x))|} \le \frac{1}{|(1-\cos(\delta)|}$.
Thereby for $x \in [\delta, 2\pi-\delta]$

$$|s_N(x) - h(x)| < \frac{\varepsilon}{|(1-\cos(x))|} \le \frac{\varepsilon}{|(1-\cos(\delta)|} \tag{1.97}$$

and this implies that in $[\delta, 2\pi-\delta]$ $s_n \longrightarrow h$ pointwise and more than pointwise: the convergence is uniform since the latter inequality does not depend on x.
To conclude the proof of the statement one has to prove that $s_n(0) \longrightarrow \frac{h(0^+)+h(0^-)}{2}$ and that $s_n(2\pi) \longrightarrow \frac{h(2\pi^+)+h(2\pi^-)}{2}$: for $x \in (0,2\pi)$ h is continuous in x, thus $\frac{h(x^+)+h(x^-)}{2} = \frac{h(x)+h(x)}{2} = h(x)$, and $s_n(x) \longrightarrow h(x)$ has been just proved.

Easily, $h(0^+) = h(2\pi^+) = \frac{\pi}{2}$ and $h(0^-) = h(2\pi^-) = -\frac{\pi}{2}$, hence $\frac{h(0^+)+h(0^-)}{2} = \frac{h(2\pi^+)+h(2\pi^-)}{2} = 0$ and the statement is proved since $\forall n \in \mathbb{N}\ s_n(0) = \sum_{k=1}^{n} \frac{\sin(k0)}{k} = 0$ and $s_n(2\pi) = \sum_{k=1}^{n} \frac{\sin(2k\pi)}{k}$.

Example 1.1.2

Let $f \in C([0,2\pi])$ be such that $f(x) = f(0) + \int_0^x g(y)\mathrm{d}y$ for a function $g \in L^2(0,2\pi)$. Notice that, comparing to Example 1.0.3, just one hypothesis has been dropped. Notice that $f(0^+) = f(0)$ and $f(0^-) = f(0)$.
Extend f to the whole real line by the procedure stated in the proof of Theorem 1.1.2. Consider the function

$$h(x) = \begin{cases} \frac{\pi-x}{2} & \text{if } x \in \mathbb{R} \setminus 2\pi\mathbb{Z} \\ 0 & \text{if } x \in 2\pi\mathbb{Z} \end{cases} \tag{1.98}$$

Define $\widetilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$

$$\widetilde{f}(x) := \begin{cases} f(x) - \frac{f(0^+)-f(0^-)}{\pi} h(x) & \text{if} \quad x \in \mathbb{R} \setminus 2\pi\mathbb{Z} \\ \frac{f(0^+)+f(0^-)}{2} & \text{if} \quad x \in 2\pi\mathbb{Z} \end{cases} \tag{1.99}$$

$\widetilde{f} \in C(\mathbb{R})$, in particular $\widetilde{f} \in C([0,2\pi])$. Indeed, for $x \in \mathbb{R} \setminus 2\pi\mathbb{Z}$ $\widetilde{f}$ is trivially continuous, whereas for $2m\pi$ $(m \in \mathbb{Z})$

$$\lim_{x \to 2m\pi^+} \widetilde{f} = f(2m\pi^+) - \frac{f(0^+)-f(0^-)}{\pi} h(2m\pi^+) = f(0^+) - \frac{f(0^+)f(0^-)}{\pi} \frac{\pi}{2} = \frac{f(0^+)+f(0^-)}{2}$$
$$\lim_{x \to 2m\pi^-} \widetilde{f} = f(2m\pi^-) - \frac{f(0^+)-f(0^-)}{\pi} h(2m\pi^-) = f(0^-) - \frac{f(0^+)f(0^-)}{\pi} \left(-\frac{\pi}{2}\right) = \frac{f(0^+)+f(0^-)}{2}$$

Moreover, for $x(0,2\pi)$

$$\begin{aligned} \widetilde{f}(x) &= f(x) - \frac{f(0^+)-f(0^-)}{\pi} \frac{\pi-x}{2} \\ &= f(0) + \int_0^x g(y)\mathrm{d}y - \frac{f(0^+)-f(0^-)}{\pi} \left(\frac{\pi}{2} - \int_0^x \frac{1}{2}\mathrm{d}y \right) \\ &= \overbrace{\frac{f(0^+)+f(0^-)}{2}}^{\widetilde{f}(0)} + \int_0^x \overbrace{\left(g(y) + \frac{f(0^+)-f(0^-)}{2\pi} \right)}^{\widetilde{g}(x)} \mathrm{d}y \end{aligned} \tag{1.100}$$

Notice that

$$\int_0^{2\pi} \widetilde{g}(y)\mathrm{d}y = f(2\pi) - f(0) + f(0^+) - f(0^-) = 0 \tag{1.101}$$

And then obviously 1.100 holds for $x \in [0,2\pi]$. Moreover

$$\int_0^{2\pi} \widetilde{g}^2(y)\mathrm{d}y = \int_0^{2\pi} \left(g^2(y) + 2\frac{f(0^+)-f(0^-)}{2\pi} g(y) + \frac{(f(0^+)-f(0^-))^2}{4\pi^2} \right) \mathrm{d}y$$
$$= \int_0^{2\pi} g^2(y)\mathrm{d}y + 2\frac{f(0^+)-f(0^-)}{2\pi} \int_0^{2\pi} g(y)\mathrm{d}y + \frac{(f(0^+)-f(0^-))^2}{2\pi} \qquad \in \mathbb{R}$$

since $g \in L^2(0,2\pi)$ and therefore also $g \in L^2(0,2\pi) \subset L^1(0,2\pi)$. This means that $\widetilde{f} \in L^2(0,2\pi)$.
$\widetilde{f}$ satisfies the hypotheses of Example 1.0.3, hence for $x \in [0,2\pi]$ the sequence $\{\widetilde{s}_n\}_n$ converges uniformly in $[0,2\pi]$. Let $\{h_n\}_n$, $\{s_n\}_n$ denote the sequences of functions associated to h and f (as functions on $[0,2\pi]$) by means of Fourier coefficients.
Fix $0 < \delta\pi$. Notice that in $[\delta, 2\pi - \delta]$

$$f(x) = \widetilde{f}(x) + \frac{f(0^+)+f(0^-)}{2} h(x) \tag{1.102}$$

Thanks to Example 1.1.1, by the linearity of the integral one can easily check that

$$s_n = \widetilde{s}_n + \frac{f(0^+)+f(0^-)}{2} h_n \overset{n\to\infty}{\longrightarrow} f \qquad \text{uniformly in } [\delta, 2\pi-\delta] \tag{1.103}$$

Moreover, also by Example 1.1.1,

$$s_n(x) = \widetilde{s}_n(x) + \frac{f(0^+)+f(0^-)}{2} h_n(x) \overset{n\to\infty}{\longrightarrow} \frac{f(x^+)+f(x^-)}{2} \qquad \text{for every } x \in [0,2\pi] \tag{1.104}$$

1.2 A classical application of Fourier series

Let $R \in \mathbb{R}$, $R > 0$ and

$$B := \{(x,y) \in \mathbb{R}^2 : \ |(x,y)| < R\}, \qquad \partial B := \{(x,y) \in \mathbb{R}^2 : \ |(x,y)| = R\} \tag{1.105}$$

Consider a continuous function $f : \ \partial B \longrightarrow \mathbb{R}$. Then a classical problem in physics is: find a function $u : \ \overline{B} \longrightarrow \mathbb{R}$, $u \in C^2(\overline{B})$, such that

$$\begin{cases} \Delta u(x,y) = 0 & \text{if } (x,y) \in B \\ u(x,y) = f(x,y) & \text{if } (x,y) \in \partial B \end{cases} \tag{1.106}$$

(the condition $\Delta u = 0$ implies that u has to be an harmonic function.) Because of the simmetry of the domain, it may be useful to switch coordinates:

$$\begin{cases} x = r\cos\vartheta \\ y = r\sin\vartheta \end{cases} \tag{1.107}$$

and define $v(r,\vartheta) := u(r\cos\vartheta, r\sin\vartheta)$. It is necessary to translate $\Delta u = 0$ in terms of v, r and ϑ, the function $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ has to be written as a function of r and θ.

Exercise 1.2.1 *Show that* $\frac{\partial^2 u}{\partial x^2}(r,\vartheta) + \frac{\partial^2 u}{\partial y^2}(r,\vartheta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)(r,\vartheta) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \vartheta^2}(r,\vartheta)$.

Solution of Exercise 1.2.1.

$$\frac{\partial v}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\vartheta + \frac{\partial u}{\partial y}\sin\vartheta$$

Then

$$\begin{aligned}
\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}\Big) &= \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial x} r\cos\vartheta\Big) + \frac{\partial}{\partial r}\Big(\frac{\partial u}{\partial y} r\sin\vartheta\Big) \\
&= \Big(\frac{\partial}{\partial r}\frac{\partial u}{\partial x}\Big) r\cos\vartheta + \frac{\partial u}{\partial x}\cos\vartheta + \Big(\frac{\partial}{\partial r}\frac{\partial u}{\partial y}\Big) r\sin\vartheta + \frac{\partial u}{\partial y}\sin\vartheta \\
&= \frac{\partial^2 u}{\partial x^2} r\cos^2\vartheta + \frac{\partial^2 u}{\partial x\partial y} r\sin\vartheta\cos\vartheta + \frac{\partial u}{\partial x}\cos\vartheta + \frac{\partial^2 u}{\partial x\partial y} r\sin\vartheta\cos\vartheta + \frac{\partial^2 u}{\partial y^2} r\sin^2\vartheta + \frac{\partial u}{\partial y}\sin\vartheta \\
&= \frac{\partial^2 u}{\partial x^2} r\cos^2\vartheta + \frac{\partial^2 u}{\partial y^2} r\sin^2\vartheta + 2\frac{\partial^2 u}{\partial x\partial y} r\sin\vartheta\cos\vartheta + \frac{\partial u}{\partial x}\cos\vartheta + \frac{\partial u}{\partial y}\sin\vartheta
\end{aligned}$$

Moreover

$$\frac{\partial v}{\partial \vartheta} = -\frac{\partial u}{\partial x} r\sin\vartheta + \frac{\partial u}{\partial y} r\cos\vartheta$$

Then

$$\begin{aligned}
\frac{\partial v}{\partial^2 \vartheta} &= \frac{\partial}{\partial \vartheta}\Big(-\frac{\partial u}{\partial x} r\sin\vartheta\Big) + \frac{\partial}{\partial \vartheta}\Big(\frac{\partial u}{\partial y} r\cos\vartheta\Big) \\
&= -\Big(-\frac{\partial^2 u}{\partial x^2} r\sin\vartheta + \frac{\partial^2 u}{\partial x\partial y} r\cos\vartheta\Big) r\sin\vartheta - \frac{\partial u}{\partial x} r\cos\vartheta + \\
&\quad + \Big(-\frac{\partial^2 u}{\partial x\partial y} r\sin\vartheta + \frac{\partial^2 u}{\partial y^2} r\cos\vartheta\Big) r\cos\vartheta - \frac{\partial u}{\partial y} r\sin\vartheta \\
&= \frac{\partial^2 u}{\partial x^2} r^2\sin^2\vartheta + \frac{\partial^2 u}{\partial y^2} r^2\cos^2\vartheta - \frac{\partial u}{\partial x} r\cos\vartheta - 2\frac{\partial^2 u}{\partial x\partial y} r\sin\vartheta\cos\vartheta - \frac{\partial u}{\partial y} r\sin\vartheta
\end{aligned}$$

Combining the above calculations,

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}(r,\vartheta)\Big)+\frac{1}{r^2}\frac{\partial v}{\partial \vartheta^2}(r,\vartheta)=$$
$$=\frac{1}{r}\Big(\frac{\partial^2 u}{\partial x^2}r\cos^2\vartheta+\frac{\partial^2 u}{\partial y^2}r\sin^2\vartheta+2\frac{\partial^2 u}{\partial x\partial y}r\sin\vartheta\cos\vartheta+\frac{\partial u}{\partial x}\cos\vartheta+\frac{\partial u}{\partial y}\sin\vartheta\Big)+$$
$$+\frac{1}{r^2}\Big(\frac{\partial^2 u}{\partial x^2}r^2\sin^2\vartheta+\frac{\partial^2 u}{\partial y^2}r^2\cos^2\vartheta-\frac{\partial u}{\partial x}r\cos\vartheta-2\frac{\partial^2 u}{\partial x\partial y}r\sin\vartheta\cos\vartheta-\frac{\partial u}{\partial y}r\sin\vartheta\Big)=$$
$$=\frac{\partial^2 u}{\partial x^2}(r\cos\vartheta,r\sin\vartheta)+\frac{\partial^2 u}{\partial y^2}(r\cos\vartheta,r\sin\vartheta)$$

Hence the previous problem can be expressed as

$$\begin{cases}\frac{\partial}{\partial r}\Big(r\frac{\partial v}{\partial r}(r,\vartheta)\Big)+\frac{1}{r}\frac{\partial v}{\partial\vartheta^2}(r,\vartheta)=0\\ v(R,\vartheta)=f(\vartheta)\end{cases}\tag{1.108}$$

The classical approach to solve the problem is based on the separation of variables method. Suppose the existence of a solution v of the form

$$v(r,\vartheta)=\varphi(r)\psi(\vartheta)\tag{1.109}$$

($\varphi,\psi\in C^2$ on the respective domains). This way, the equation in B becomes

$$\psi(\vartheta)\big(r\varphi'(r)\big)'+\frac{1}{r}\varphi(r)\psi''(\vartheta)=0\tag{1.110}$$

If for $(r_0,\vartheta_0)\in B$ is such that $v(r_0,\vartheta_0)\neq 0$, then in a neighbourhood of (r_0,ϑ_0), dividing, the above equation is equivalent to

$$\frac{\big(r\varphi'(r)\big)'}{\varphi(r)}+\frac{\psi''(\vartheta)}{\psi(\vartheta)}=0\tag{1.111}$$

The two summands depend on r and ϑ separately, there exists $c\in\mathbb{R}$ such that the equation is equivalent to the system

$$\begin{cases}\frac{\big(r\varphi'(r)\big)'}{\varphi(r)}=c\\ \frac{\psi''(\vartheta)}{\psi(\vartheta)}=-c\end{cases}\qquad\begin{cases}r^2\varphi''+r\varphi'-c\varphi=0\\ \psi''+c\psi=0\end{cases}\tag{1.112}$$

1. If $c=0$, $\psi''=0$ implies that ψ is linear, but since $v(R,\vartheta)=f(\vartheta)$ on ∂B and f is continuous, it has to be $\psi(0)=\psi(2\pi)$, hence $\psi(\vartheta)=A_0$ with $A_0\in\mathbb{R}$.
   $r^2\varphi''+r\varphi'=0$, thus $r\varphi''+\varphi'$. Define $\alpha=\varphi'$, then $r\alpha'+\alpha=0$. Suppose that $\exists(r_0,\vartheta_0)\in B$ such that $\alpha(r_0,\vartheta_0)\neq 0$; then, for the regularity of v, α does not assume the value 0 in an open neighbourhood of (r_0,ϑ_0) in which

$$\frac{\alpha'}{\alpha} = -\frac{1}{r} \quad \Rightarrow \quad \log\alpha = -\log r + C,\ C \in \mathbb{R}$$

$$\Rightarrow \quad \varphi' = \alpha = \frac{e^C}{r},\ C \in \mathbb{R} \quad \Rightarrow \quad \varphi(r) = e^C \log r + D,\ C, D \in \mathbb{R}$$

But then, the solution can be extended. Such a φ is not defined for $r = 0$, hence α must be 0 in a closed ball $\overline{B((0,0),R_0)}$ with R_0 such that $e^C \log R_0 + D = 0$ and $(r_0, \vartheta_0) \notin \overline{B((0,0),R_0)}$. α must be C^1 since φ is C^2, thus $\frac{d\alpha}{dr}$ must be continuous. For $R_0 < r < R$ $\frac{d\alpha}{dr}(r) = -\frac{e^C}{r^2}$, and the requirement that α' exists continuous implies

$$0 = \lim_{r \to R_0^-} \alpha'(r) = \lim_{r \to R_0^+} \alpha'(r) = -\frac{e^C}{R_0^2}$$

This is absurd, and it follows that α must be 0 in B, hence in B $\varphi(r) = C$ for $C \in \mathbb{R}$.
From the previous discussion it follows that if $c = 0$, then in B $v(r, \vartheta) = \frac{A_0}{2}$, with $A_0 \in \mathbb{R}$.
2. If $c - 0$, $\psi'' - \omega^2 \psi = 0$ with $\omega \in \mathbb{R} \setminus \{0\}$. The general solution for an ODE of this kind is $\psi(\vartheta) = Ae^{\omega\vartheta} + Be^{-\omega\vartheta}$ for $A, B \in \mathbb{R}$. But then the conditions $\psi(0) = \psi(2\pi)$ and $\psi'(0) = \psi'(2\pi)$ yield, considering that $\psi'(\vartheta) = \omega(Ae^{\omega\vartheta} - Be^{-\omega\vartheta})$,

$$\begin{cases} A + B = Ae^{2\pi\omega} + Be^{-2\pi\omega} \\ \omega(A - B) = \omega(Ae^{2\pi\omega} - Be^{-2\pi\omega}) \end{cases} \quad \Rightarrow \quad \begin{cases} A = 0 \\ B = 0 \end{cases}$$

Hence for $c < 0$ the only possible solution is the null function.
3. If $c > 0$, then $\psi'' + \omega^2 \psi = 0$ with $\omega \in \mathbb{R} \setminus \{0\}$. From general ODE theory, solutions can be written as $\psi(\vartheta) = A\cos(\omega\vartheta) + B\sin(\omega\vartheta)$ with $A, B \in \mathbb{R}$. Such a ψ has period $\frac{2\pi}{\omega}$. Since an integer multiple of the period must be 2π, then all the solution to the particular problem can be written as $\psi_n(\vartheta) = A_n\cos(n\vartheta) + B_n\sin(n\vartheta)$ with $A, B \in \mathbb{R}$, $n \in \mathbb{N}$.
The equation $r^2\varphi'' + r\varphi' - n^2\varphi = 0$ for $n \in \mathbb{N}$ is e linear second order ODE with nonconstant coefficients, hence the space of solutions is a vector space of dimension 2. Notice that $\varphi_1(r) := r^n$, $\varphi_2(r) := r^{-n}$ are both solutions and they are linearly independent, thus the general solution of the equation can be written as $\varphi(r) = A\varphi_1(r) + B\varphi_2(r)$, with $A, B \in \mathbb{R}$. φ_2 has a singularity in 0, $B = 0$.

Then, from the above discussion, a solution v of $\Delta v = 0$, obtained using the separation of variables method, is

$$v_n(r, \vartheta) = \frac{r^n}{R^n}(A_n\cos(n\vartheta) + B_n\sin(n\vartheta)), \qquad n \in \mathbb{N} \cup \{0\} \tag{1.113}$$

Notice that the operator Δ is linear. Hence, if u_1, u_2 are solutions of $\Delta v = 0$ and $\lambda \in \mathbb{R}$, also $\Delta(u_1 + \lambda u_2) = \Delta u_1 + \lambda\Delta u_2 = 0$, i.e. $u_1 + \lambda u_2$ is still a solution of $\Delta v = 0$. Therefore $\forall n \in \mathbb{N} \cup \{0\}$ choosing a function v_k for each $0 \le k \le n$ one gets that $\sum_{k=0}^{n} v_k(r, \vartheta)$ is a solution of $\Delta v = 0$.

Chapter 2
Fourier Transform

Definition 2.0.1 *Consider $f \in L^1(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$. The Fourier transform of f is the function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ defined, for $\xi \in \mathbb{R}$, as*

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-ix\xi} f(x) \mathrm{d}x \tag{2.1}$$

Since $f \in L^1(\mathbb{R})$, $\widehat{f}$ is well-defined. Indeed, $|e^{-ix\xi} f(x)| = |e^{-ix\xi}||f(x)| = |f(x)|$, therefore $\int_{\mathbb{R}} |e^{-ix\xi} f(x)| \mathrm{d}x = \int_{\mathbb{R}} |f(x)| \mathrm{d}x \in \mathbb{R}$.

Exercise 2.0.2 *Show that $\int_0^{+\infty} e^{-(\lambda+i\xi)x} \mathrm{d}x = \frac{1}{\lambda+i\xi}$ for $\lambda, \xi \in \mathbb{R}$ and $\lambda > 0$.*

Solution of Exercise 2.0.2.

$$\int_0^{+\infty} e^{-(\lambda+i\xi)x} \mathrm{d}x = \lim_{L \to +\infty} \left[-\frac{e^{-(\lambda+i\xi)x}}{\lambda+i\xi} \right]_0^L = \frac{1}{\lambda+i\xi}$$

Example 2.0.1

Let $\lambda \in \mathbb{R}$, $\lambda > 0$. Consider $f(x) := e^{-\lambda|x|} \in L^1(\mathbb{R})$. Then

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{\mathbb{R}} e^{-ix\xi} e^{-\lambda|x|} \mathrm{d}x = \int_{-\infty}^{0} e^{(\lambda-i\xi)x} \mathrm{d}x + \int_0^{+\infty} e^{-(\lambda+i\xi)x} \mathrm{d}x \\
&= \int_0^{+\infty} e^{-(\lambda-i\xi)x} \mathrm{d}x + \int_0^{+\infty} e^{-(\lambda+i\xi)x} \mathrm{d}x \overset{\text{Exercise 2.0.2}}{=} \frac{1}{\lambda-i\xi} + \frac{1}{\lambda+i\xi} = \frac{2\lambda}{\lambda^2+\xi^2}
\end{aligned}$$

Note that the Fourier transform $\widehat{f}$ in this example is real-valued. This could have been foreseen noticing that the function $e^{-\lambda|x|}$ is even, thereby the imaginary part of $e^{-ix\xi} e^{-\lambda|x|}$ is odd, implying that its integral over $\mathbb{R}$ is 0.

Exercise 2.0.3 *Show again that $\widehat{e^{-\lambda|x|}}(\xi) = \frac{2\lambda}{\lambda^2+\xi^2}$ by means of the observation*

$$\widehat{e^{-\lambda|x|}}(\xi) = \int_{\mathbb{R}} \cos(x\xi) e^{-\lambda|x|} \mathrm{d}x = 2 \int_0^{2\pi} \cos(x\xi) e^{-\lambda x} \mathrm{d}x \tag{2.2}$$

Exercise 2.0.4 *Let $\lambda \in \mathbb{R}$, $\lambda > 0$ and consider the function over $\mathbb{R}$ $f(x) := e^{-\lambda x^2}$. Prove that*

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} e^{-\lambda x^2} \mathrm{d}x = \sqrt{\frac{\pi}{\lambda}}\, e^{-\frac{\xi^2}{4\lambda}} \tag{2.3}$$

A first solution of Exercise 2.0.4.

$$\begin{aligned}\varphi(\xi) &:= \int_{\mathbb{R}} e^{-ix\xi} e^{-\lambda x^2} \mathrm{d}x = \int_{\mathbb{R}} \cos(x\xi) e^{-\lambda x^2} \mathrm{d}x - i \int_{\mathbb{R}} \overbrace{\sin(x\xi)}^{odd} \overbrace{e^{-\lambda x^2}}^{even} \mathrm{d}x \\ &= \int_{\mathbb{R}} \cos(x\xi) e^{-\lambda x^2} \mathrm{d}x\end{aligned}$$

$$\varphi'(\xi) = -\int_{\mathbb{R}} x \sin(x\xi) e^{-\lambda x^2} \mathrm{d}x = \frac{1}{2\lambda} \int_{\mathbb{R}} \sin(x\xi) \frac{de^{-\lambda x^2}}{dx} \mathrm{d}x = -\frac{\xi}{2\lambda} \int_{\mathbb{R}} \cos(x\xi) e^{-\lambda x^2} \mathrm{d}x$$

Hence $\varphi'(\xi) = -\frac{\xi}{2\lambda}\varphi(\xi)$, and the solution of this ODE is, for $\in \mathbb{R}$, $\varphi(\xi) = ae^{-\frac{\xi^2}{2\lambda}}$. Since it is well-known that

$$\int_{\mathbb{R}} e^{-x^2} \mathrm{d}x = \sqrt{\pi} \tag{2.4}$$

for $\xi = 0$, one gets

$$a = \varphi(0) = \int_{\mathbb{R}} e^{-\lambda x^2} \mathrm{d}x \overset{y \overset{s}{=} \sqrt{\lambda} x}{=} \frac{1}{\sqrt{\lambda}} \int_{\mathbb{R}} e^{-y^2} \mathrm{d}y = \sqrt{\frac{\pi}{\lambda}} \tag{2.5}$$

A second solution of Exercise 2.0.4.

$$\begin{aligned}\varphi(\xi) &:= \int_{\mathbb{R}} e^{-ix\xi} e^{-\lambda x^2} \mathrm{d}x = \int_{\mathbb{R}} \cos(x\xi) e^{-\lambda x^2} \mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\sum_{k=0}^{\infty} (-1)^k \frac{(x\xi)^{2k}}{(2k)!} \right) e^{-\lambda x^2} \mathrm{d}x = \sum_{k=0}^{\infty} (-1)^k \frac{(\xi)^{2k}}{(2k)!} \int_{\mathbb{R}} x^{2k} e^{-\lambda x^2} \mathrm{d}x\end{aligned}$$

(it can be proved that the exchange of sum and integral is allowed). Define for $k \in \mathbb{N} \cup \{0\}$,

$$I_k := \int_{\mathbb{R}} x^{2k} e^{-\lambda x^2} \mathrm{d}x \overset{\ddagger}{=} \sqrt{\frac{\pi}{\lambda}} \frac{1}{(2\lambda)^k} \prod_{m=1}^{k} (2m-1) \tag{2.6}$$

$\ddagger$ can be proved by induction.

$$I_0 = \int_{\mathbb{R}} e^{-\lambda x^2} \mathrm{d}x \overset{2.5}{=} \sqrt{\frac{\pi}{\lambda}}$$

Suppose $k \geq 1$, and that $\ddagger$ holds for $0 \leq j \leq k-1$. Then

$$I_k = \int_{\mathbb{R}} x^{2k} e^{-\lambda x^2} \mathrm{d}x = -\frac{1}{2\lambda}\int_{\mathbb{R}} x^{2k-1}\frac{de^{-\lambda x^2}}{dx}\mathrm{d}x \overset{\text{by parts}}{=} \frac{2k-1}{2\lambda}\int_{\mathbb{R}} x^{2(k-1)}e^{-\lambda x^2}\mathrm{d}x$$
$$= \frac{2k-1}{2\lambda} I_{k-1} = \frac{2k-1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}\frac{1}{(2\lambda)^{k-1}}\prod_{m=1}^{k-1}(2m-1) = \sqrt{\frac{\pi}{\lambda}}\frac{1}{(2\lambda)^k}\prod_{m=1}^{k}(2m-1)$$

Hence

$$\varphi(\xi) = \sum_{k=0}^{\infty}(-1)^k\frac{(\xi)^{2k}}{(2k)!}I_k = \sqrt{\frac{\pi}{\lambda}}\sum_{k=0}^{\infty}(-1)^k\frac{(\xi)^{2k}}{(2k)!}\frac{1}{(2\lambda)^k}\prod_{m=1}^{k}(2m-1)$$
$$= \sqrt{\frac{\pi}{\lambda}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{\xi^2}{2\lambda}\right)^k\prod_{m=1}^{k}\frac{1}{2m} = \sqrt{\frac{\pi}{\lambda}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\xi^2}{4\lambda}\right)^k = \sqrt{\frac{\pi}{\lambda}}e^{-\frac{\xi^2}{4\lambda}}$$

A third solution of Exercise 2.0.4.

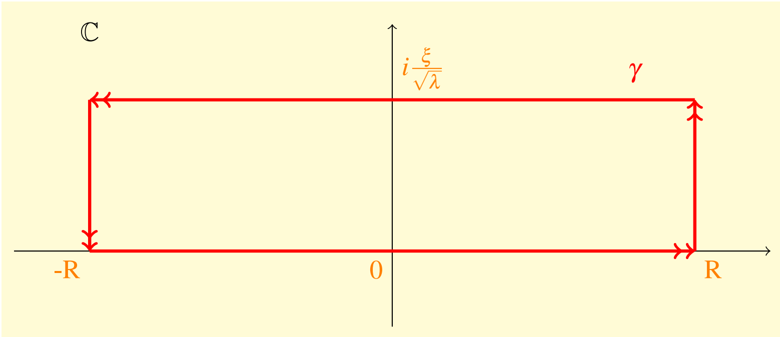
$$\int_{\mathbb{R}} e^{-(\lambda x^2+ix\xi)}\mathrm{d}x = \int_{\mathbb{R}} e^{-\left[\left(\sqrt{\lambda}x+\frac{i\xi}{\sqrt{\lambda}}\right)^2+\frac{\xi^2}{4\lambda}\right]}\mathrm{d}x = e^{\frac{\xi^2}{4\lambda}}\int_{\mathbb{R}} e^{-\left(\sqrt{\lambda}x+\frac{i\xi}{\sqrt{\lambda}}\right)^2}\mathrm{d}x$$
$$\overset{y\overset{s}{=}\sqrt{\lambda}x}{=} \frac{e^{\frac{\xi^2}{4\lambda}}}{\sqrt{\lambda}}\int_{\mathbb{R}} e^{-\left(y+\frac{i\xi}{\sqrt{\lambda}}\right)^2}\mathrm{d}y = \frac{e^{\frac{\xi^2}{4\lambda}}}{\sqrt{\lambda}}\lim_{R\to+\infty}\int_{-R}^{R} e^{-\left(y+\frac{i\xi}{\sqrt{\lambda}}\right)^2}\mathrm{d}y$$

Now recall that if $\Omega \subset \mathbb{C}$ is a simply connected open set and $g : \Omega \longrightarrow \mathbb{C}$ is a holomorphic function, then for any closed path $\gamma \subset \Omega$

$$\int_{\gamma} g(z)\mathrm{d}z = 0 \tag{2.7}$$

This result provides a way to compute the needed integral. In the following computation the assumption is $\xi \geq 0$. For $\lambda < 0$ the computation is analogous.
The function $g : \mathbb{C} \longrightarrow \mathbb{C}$ $g(z) := e^{-z^2}$ is holomorphic in the simply connected and open set $\mathbb{C}$. Then, for $R \in \mathbb{R}_+$, consider the closed path $\gamma\colon [0,1] \longrightarrow \mathbb{C}$ defined as

$$\gamma(t) := \begin{cases} R + it\frac{\xi}{\sqrt{\lambda}} & \text{if } t \in \left[0, \frac{1}{4}\right] \\ (3-8t)R + i\frac{\xi}{\sqrt{\lambda}} & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ -R + i\frac{\xi}{\sqrt{\lambda}}(3-4t) & \text{if } t \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ (6t-7)R & \text{if } t \in \left[\frac{3}{4}, 1\right] \end{cases} \tag{2.8}$$



Hence

$$0=\int_\gamma e^{-z^2}\mathrm{d}z=\int_{-R}^{R}e^{-z^2}\mathrm{d}z+\int_{R}^{R+i\frac{\xi}{\sqrt{\lambda}}}e^{-z^2}\mathrm{d}z+\int_{R+i\frac{\xi}{\sqrt{\lambda}}}^{-R+i\frac{\xi}{\sqrt{\lambda}}}e^{-z^2}\mathrm{d}z+\int_{-R+i\frac{\xi}{\sqrt{\lambda}}}^{-R}e^{-z^2}\mathrm{d}z$$
$$=\int_{-R}^{R}e^{-y^2}\mathrm{d}y+i\int_0^{\frac{\xi}{\sqrt{\lambda}}}e^{-(R+iy)^2}\mathrm{d}y+\int_R^{-R}e^{-(y+i\frac{\xi}{\sqrt{\lambda}})^2}\mathrm{d}y+i\int_{\frac{\xi}{\sqrt{\lambda}}}^{0}e^{-(-R+iy)^2}\mathrm{d}y \tag{2.9}$$

And

$$\left|i\int_0^{\frac{\xi}{\sqrt{\lambda}}}e^{-(R+iy)^2}\mathrm{d}y\right|\le\int_0^{\frac{\xi}{\sqrt{\lambda}}}\left|e^{-(R^2-y^2+2iyR)}\right|\mathrm{d}y=\int_0^{\frac{\xi}{\sqrt{\lambda}}}e^{y^2-R^2}\mathrm{d}y$$
$$\le\int_0^{\frac{\xi}{\sqrt{\lambda}}}e^{\frac{\xi^2}{\lambda}-R^2}\mathrm{d}y=\frac{\xi}{\sqrt{\lambda}}e^{\frac{\xi^2}{\lambda}-R^2}\xrightarrow{R\to+\infty}0$$

$$\left|i\int_{\frac{\xi}{\sqrt{\lambda}}}^{0}e^{-(R+iy)^2}\mathrm{d}y\right|\le\int_0^{\frac{\xi}{\sqrt{\lambda}}}\left|e^{y^2-R^2}\right|\mathrm{d}y=\frac{\xi}{\sqrt{\lambda}}e^{\frac{\xi^2}{\lambda}-R^2}\xrightarrow{R\to+\infty}0$$

The conclusion of the exercise follows taking the limit for $R\to+\infty$ in 2.9:

$$\int_{-\infty}^{+\infty}e^{-(y+i\frac{\xi}{\sqrt{\lambda}})^2}\mathrm{d}y=\int_{-\infty}^{+\infty}e^{-y^2}\mathrm{d}y=\sqrt{\pi} \tag{2.10}$$

Example 2.0.2

Let $a\in\mathbb{R}$, $a>0$ and $f(x):=1$ if $x\in[-a,a]$, $f(x):=0$ if $x\in\mathbb{R}\setminus[-a,a]$. Then for $\xi\neq 0$

$$\widehat{f}(\xi)=\int_{-a}^{a}e^{-ix\xi}\mathrm{d}x=\int_{-a}^{a}\cos(x\xi)\mathrm{d}x=\left[\frac{\sin(x\xi)}{\xi}\right]_{-a}^{a}=2\,\frac{\sin(a\xi)}{\xi} \tag{2.11}$$

Whereas for $\xi=0$

$$\widehat{f}(\xi) = \int_{-a}^{a} e^0 \mathrm{d}x = 2a \tag{2.12}$$

Let

$$C_0(\mathbb{R}) := \{g \in \mathbb{C}^{\mathbb{R}} : g \text{ is continuos and } \lim_{\xi \to -\infty} g(\xi) = \lim_{\xi \to +\infty} g(\xi) = 0\} \tag{2.13}$$

and define $\|\cdot\|_\infty : C_0(\mathbb{R}) \longrightarrow \mathbb{R}$ as

$$\|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)| \tag{2.14}$$

Exercise 2.0.5 *(i) $C_0(\mathbb{R})$ is a vector space over $\mathbb{C}$.*
(ii)$\|\cdot\|_\infty$ is a norm in $C_0(\mathbb{R})$.
(iii)$(C_0(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space.

Given $f \in L^1(\mathbb{R})$, $\widetilde{f}$ enjoys the following properties

1. $\widetilde{f}$ is continuous.
2. $\widetilde{f} \in C_0(\mathbb{R})$.

A linear function between Banach spaces is called linear operator

Exercise 2.0.6 *Let $f, g \in L^1(\mathbb{R})$, $\lambda \in \mathbb{R}$. Then $\widehat{(f + \lambda g)} = \widehat{f} + \lambda \widehat{g}$, i.e.$\widehat{\ }$: $(L^1(\mathbb{R}), \|\cdot\|_{L^1}) \longrightarrow (C_0(\mathbb{R}), \|\cdot\|_\infty)$ is a linear operator.*

Definition 2.0.2 *Consider $f \in L^1(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{C}$. The inverse Fourier transform of f is the function $\check{f} : \mathbb{R} \to \mathbb{C}$ defined, for $\xi \in \mathbb{R}$, as*

$$\check{f}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} f(x) \mathrm{d}x \tag{2.15}$$

Notice that

$$\check{f}(\xi) = \frac{1}{2\pi} \widetilde{f}(-\xi) \tag{2.16}$$

Hence $\check{f} \in C_0(\mathbb{R})$ and $\check{\ }$: $(L^1(\mathbb{R}), \|\cdot\|_{L^1}) \longrightarrow (C_0(\mathbb{R}), \|\cdot\|_\infty)$ is a linear operator.

Remark 3 $C_0(\mathbb{R}) \nsubseteq L^1(\mathbb{R})$.

For instance

$$g(x) := \begin{cases} 1 & \text{if } x \in [-1, 1] \\ \frac{1}{\sqrt{|x|}} & \text{if } x \in \mathbb{R} \setminus [-1, 1] \end{cases} \tag{2.17}$$

belongs to $C_0(\mathbb{R})$, but it does no belong to $L^1(\mathbb{R})$.

Exercise 2.0.7 *For $a \in \mathbb{R}$, the function*

$$h(x) := \begin{cases} 2\frac{\sin ax}{x} & \text{if } x \in \mathbb{R}, x \neq 0 \\ 2a & \text{if } x = 0 \end{cases} \tag{2.18}$$

does not belong to $L^1(\mathbb{R})$, but it belongs to $C_0(\mathbb{R})$.

Exercise 2.0.8 *Since, for* $\lambda \in \mathbb{R}$, $\lambda > 0$ $\widehat{e^{-\lambda|x|}}, \widehat{e^{-\lambda x^2}} \in L^1(\mathbb{R})$, *compute the inverse Fourier transform of both* $\widehat{e^{-\lambda|x|}}$ *and* $\widehat{e^{-\lambda x^2}}$.

Remark 4 *Let* $f \in L^1(\mathbb{R})$. *Then* $\|\widehat{f}\|_\infty \leq \|f\|_{L^1}$, *that is to say,* $\widehat{}$ *is a bounded linear operator, and also* $\check{}$ *is a bounded linear operator.*

Indeed,

$$\|\widehat{f}\|_\infty = \sup_{\xi\in\mathbb{R}} \left| \int_{\mathbb{R}} e^{-ix\xi} f(x)\mathrm{d}x \right| \leq \sup_{\xi\in\mathbb{R}} \int_{\mathbb{R}} |e^{-ix\xi} f(x)|\mathrm{d}x = \sup_{\xi\in\mathbb{R}} \int_{\mathbb{R}} |f(x)|\mathrm{d}x$$
$$= \sup_{\xi\in\mathbb{R}} \|f\|_{L^1} = \|f\|_{L^1}$$

and

$$\|\check{f}\|_\infty = \sup_{\xi\in\mathbb{R}} \left| \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ix\xi} f(x)\mathrm{d}x \right| = \frac{1}{2\pi} \|\widehat{f}\|_\infty \leq \frac{1}{2\pi} \|f\|_{L^1}$$

Proposition 2.0.1 *Let* $f, g \in L^1(\mathbb{R})$. *Then*

$$\int_{\mathbb{R}} f(x)\widehat{g}(x)\mathrm{d}x = \int_{\mathbb{R}} \widehat{f}(x)g(x)\mathrm{d}x \tag{2.19}$$

$$\int_{\mathbb{R}} f(x)\check{g}(x)\mathrm{d}x = \int_{\mathbb{R}} \check{f}(x)g(x)\mathrm{d}x \tag{2.20}$$

Proof. First of all, one has to show that the integrals displayed in the statement make sense. Indeed, since the Fourier transform and the inverse Fourier transform are functions in $C_0(\mathbb{R})$, they are bounded by a constant $M \in \mathbb{R}$. Thereby the integral of the module of each of those products is less or equal than M times the L^1 norm of either f or g. This forces more than the good definition of the displayed integrals, it forces the four products to be functions in L^1.
Consider the function that maps $(x, y) \in \mathbb{R}\times\mathbb{R}$ to $f(x)g(y)$. By Fubini Theorem, it is immediate that it belongs to $L^1(\mathbb{R}\times\mathbb{R})$:

$$\int\int_{\mathbb{R}\times\mathbb{R}} f(x)g(y)\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}} f(x)\mathrm{d}x \cdot \int_{\mathbb{R}} g(y)\mathrm{d}y \quad \in \mathbb{R}$$

Since for $(x, y) \in \mathbb{R}\times\mathbb{R}$ $|e^{-ixy} f(x)g(y)| = |f(x)g(y)|$, also the function $(x, y) \mapsto e^{-ixy} f(x)g(y)$ belongs to $L^1(\mathbb{R}\times\mathbb{R})$. This enables to compute the integral of the latter function, and using again Fubini Theorem

$$\int_{\mathbb{R}} f(x)\widehat{g}(x)\mathrm{d}x = \int_{\mathbb{R}} f(x)\left(\int_{\mathbb{R}} e^{-ixy} g(y)\mathrm{d}y\right)\mathrm{d}x = \int\int_{\mathbb{R}\times\mathbb{R}} f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$= \int_{\mathbb{R}} g(y)\left(\int_{\mathbb{R}} e^{-ixy} f(x)\mathrm{d}x\right)\mathrm{d}y = \int_{\mathbb{R}} \widehat{f}(x)g(x)\mathrm{d}x$$

And similarly also the second equality in the above statement can be proved.

Given $f, g \in L^1(\mathbb{R})$, the convolution product of f and g (which has already been used) is defined as, for $x \in \mathbb{R}$,

$$(\hat{f} * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y \tag{2.21}$$

Theorem 2.0.1 *Let $f, g \in L^1(\mathbb{R})$. Then $f * g \in L^1(\mathbb{R})$ and moreover*

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1} \tag{2.22}$$

Exercise 2.0.9 *Let $f, g, h \in L^1(\mathbb{R})$, $\lambda \in \mathbb{R}$. Then prove that*

1. $(\overbrace{f+g}^{\in L^1}) * h = f * h + g * h$
2. $f * g = g * f$
3. $(f * g) * h = f * (g * h)$
4. $(\lambda f) * g = \lambda (f * g)$

Definition 2.0.3 *Given a commutative ring $(R, \tilde{+}, \tilde{\times})$, $(A, +, \times)$ is called an associative $R-$algebra if*

(i) $(A, +, \times)$ is a ring
(ii) A can be endowed with a structure of $R-$module such that the ring multiplication $\times$ is R-bilinear, i.e. for $w \in R$, $\gamma, \mu \in A$ $w(\gamma * \mu) = (w\gamma) * \mu = \gamma * (w\mu)$

If $(A, +, \times)$ is an associative algebra and $(A, \|\cdot\|)$ is a Banach space, $(A, +, \times, \|\cdot\|)$ is a Banach algebra if

$$\forall \gamma, \mu \in A \qquad \|\gamma \times \mu\| \leq \|\gamma\| \, \|\mu\| \tag{2.23}$$

From Exercise 2.0.9 it is immediate that $\left(L^1(\mathbb{R}), +, *\right)$ is a ring (without unit). Since $\left(L^1(\mathbb{R}), +\right)$ is a vector space over $\mathbb{C}$, thanks to the homogeneity of the integral and to Theorem 2.0.1, it follows that $\left(L^1(\mathbb{R}), +, *, \|\cdot\|_{L^1}\right)$ is a Banach algebra without unit (the absence of unit can be related to a Gelfand Theorem). Actually, it is a commutative Banach algebra.

Corollary 2.0.1 *Let $f, g \in L^1(\mathbb{R})$. Then for $a \in \mathbb{R}$*

$$\frac{1}{2\pi}(\hat{f} * g)(a) = \widehat{(f\check{g})}(a) \tag{2.24}$$

Notice that $\psi : \mathbb{R} \longrightarrow \mathbb{C}$, $\psi(x) := e^{-ixa} f(x)$ belongs to $L^1(\mathbb{R})$. Moreover

$$\check{\psi}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixy} e^{-iya} f(y)\mathrm{d}y = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i(a-x)y} f(y)\mathrm{d}y = \frac{1}{2\pi}\widehat{f}(a-x)$$
$$\hat{\psi}(x) = \int_{\mathbb{R}} e^{-ixy} e^{-iya} f(y)\mathrm{d}y = \int_{\mathbb{R}} e^{-i(x+a)y} f(y)\mathrm{d}y = \widehat{f}(x+a)$$

Proof. Using the notation of the remark above

$$\widehat{(f\check{g})}(a) = \int_{\mathbb{R}} e^{-ixa} f(x)\check{g}(x)\mathrm{d}x \overset{\text{Proposition 2.0.1}}{=} \int_{\mathbb{R}} \check{\psi}(x)g(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(a-x)g(x)\mathrm{d}x = \frac{1}{2\pi}(\hat{f}*g)(a)$$

Remark 5 *The function $\hat{f}$ does not necessarily belong to $L^1(\mathbb{R})$ and the convolution product has been defined for functions in $L^1(\mathbb{R})$, but since $\hat{f}\in C_0(\mathbb{R})$, it is bounded. $g\in L^1(\mathbb{R})$, and therefore that "extension "of the convolution product makes sense and is an $\mathbb{R}$-valued function.*

Example 2.0.3

For $\lambda\in\mathbb{R}$, $\lambda>0$ define $f(x) := e^{-\lambda x^2}$ (see Exercise 2.0.4). Pick $a\in\mathbb{R}$

- Suppose $g\in L^1(\mathbb{R})$. Then, using Proposition 2.0.1, the remark between the statement of Corollary 2.0.1 and its proof, and then Exercise 2.0.4,

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{iax}e^{-\lambda x^2}\widehat{g}(x)\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}}\widehat{f}(x-a)g(x)\mathrm{d}x = \frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{4\lambda}}g(x)\mathrm{d}x \tag{2.25}$$

What happens when $\lambda\to 0$?

- Suppose $g\in L^1(\mathbb{R})$ to be such that also $\widehat{g}\in L^1(\mathbb{R})$. Then

$$|e^{iax}e^{-\lambda x^2}\widehat{g}(x)| = |e^{-\lambda x^2}\widehat{g}(x)| \leq |\widehat{g}(x)|$$

Thanks to the new hypothesis, considering that $e^{iax}e^{-\lambda x^2}\widehat{g}(x)\overset{\lambda\to 0}{\longrightarrow} e^{iax}\widehat{g}(x)$, using Lebesgue dominated convergence theorem

$$\lim_{\lambda\to 0}\left(\frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{4\lambda}}g(x)\mathrm{d}x\right) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iax}\widehat{g}(x)\mathrm{d}x = \check{\hat{g}}(a) \tag{2.26}$$

Exercise 2.0.10 *Define for $\lambda, a\in\mathbb{R}$, $\lambda>0$ the function over $\mathbb{R}$ $\gamma_{a,\lambda}(x) := \frac{1}{\sqrt{4\pi\lambda}}e^{-\frac{(x-a)^2}{4\lambda}}$. Prove that $\int_{\mathbb{R}}\gamma_{a,\lambda}(x)\mathrm{d}x = 1$.*

- Lastly, suppose $g\in L^1(\mathbb{R})\cap C_b(\mathbb{R})$ to be such that $\widehat{g}\in L^1(\mathbb{R})$ ($C_b(\mathbb{R})$ denotes the space of bounded continuous functions over $\mathbb{R}$). Then

$$\lim_{\lambda\to 0}\left(\frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{4\lambda}}g(x)\mathrm{d}x\right) = g(a) \tag{2.27}$$

In particular, cosidering the previous point, it follows that, given these hypotheses, $\forall a\in\mathbb{R}\quad \check{\hat{g}}(a) = g(a)$ Indeed $g\in C_b(\mathbb{R})$ implies that $\exists M\in\mathbb{R}$ such that for every $x\in\mathbb{R}$ $|g(x)|\leq M$. Moreover, $\forall\varepsilon>0\ \exists\delta_\varepsilon : \forall x,y\in\mathbb{R}\ |x-y|<\delta_\varepsilon \Rightarrow |g(x)-g(y)|<\varepsilon$. Hence,

$$\left|\frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{4\lambda}} g(x)\mathrm{d}x - g(a)\right| \overset{\text{Exercise 2.0.10}}{=} \left|\frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{4\lambda}} [g(x)-g(a)]\mathrm{d}x\right|$$
$$\leq \frac{1}{\sqrt{4\pi\lambda}}\int_{|x-a|<\delta_\varepsilon} e^{-\frac{(x-a)^2}{4\lambda}} |g(x)-g(a)|\mathrm{d}x + \frac{1}{\sqrt{4\pi\lambda}}\int_{|x-a|\geq\delta_\varepsilon} e^{-\frac{(x-a)^2}{4\lambda}} |g(x)-g(a)|\mathrm{d}x$$
$$\overset{\text{Exercise 2.0.10}}{<} 1\cdot\varepsilon + \frac{2M}{\sqrt{4\pi\lambda}}\int_{|x-a|\geq\delta_\varepsilon} e^{-\frac{(x-a)^2}{4\lambda}}\mathrm{d}x = \varepsilon + 2M\tau(\varepsilon,\lambda)$$

where $\tau(\varepsilon,\lambda) \overset{\lambda\to 0}{\longrightarrow} 0$. Hence $\forall\varepsilon > 0$

$$0 \leq \limsup_{\lambda\to 0}\left|\frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{4\lambda}} g(x)\mathrm{d}x - g(a)\right| = \varepsilon$$

thus $\exists\lim_{\lambda\to 0}\left|\frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} e^{-\frac{(x-a)^2}{4\lambda}} g(x)\mathrm{d}x - g(a)\right| = 0$ and the claim is proved.

Notice that the function $g(x) := e^{-\lambda|x|}$ satisfies the three assumptions stated in the third point of Example 2.0.3 ($\widehat{g}(\xi) = \frac{2\lambda}{\lambda^2+\xi^2}$, see Example 2.0.1), hence $\check{\hat{g}} = g$ and also $\hat{\check{g}}$. As a further consequence, one gets that

$$\frac{1}{2\pi}\int_{\mathbb{R}} e^{ix\xi}\frac{2\lambda}{\lambda^2+\xi^2}\mathrm{d}x = e^{-\lambda|x|} \tag{2.28}$$

2.1 The space $\mathscr{S}$

Define

$$\mathscr{S} := \left\{\varphi\in C^\infty(\mathbb{R}) :\ \forall\alpha,\beta\in\mathbb{N}\cup\{0\}\ \exists C_{\alpha,\beta}\in\mathbb{R}\text{ s.t. } |x^\alpha D^\beta\varphi(x)|\leq C_{\alpha,\beta}\right\} \tag{2.29}$$

$\mathscr{S}$ is called Schwartz space or space of fast decreasing functions.
Let $\varphi\in\mathscr{S}$. Then φ and all its derivatives are in $C_b(\mathbb{R})$: it follows easily choosing $\alpha = 0$ in the definition of $\mathscr{S}$. More than that, φ and all its derivatives are in $C_b(\mathbb{R})$, and it is interesting to notice the convergence to zero at infinity is faster than the convergence to zero at zero of any polynomial: for $x\neq 0$

$$|D^\beta(x)| \leq C_{\alpha,\beta}\left(\frac{1}{|x|}\right)^\alpha \tag{2.30}$$

Notice that $\mathscr{S}$ is not empty. Indeed, the function $x\mapsto 0$ belongs to $\mathscr{S}$. Less trivially, for $\lambda\in\mathbb{R}$, $\lambda > 0$, $e^{-\lambda x^2}\in\mathscr{S}$ (but e.g. $e^{-\lambda|x|}\notin\mathscr{S}$ because it is not smooth in 0).

Exercise 2.1.1 *$\mathscr{S}$ is a vector space over $\mathbb{C}$.*

Remark 6 $\mathscr{S}\subset L^1(\mathbb{R})$.

In fact, take $\varphi \in \mathscr{S}$ and consider both $\alpha = 2, \beta = 0$ and $\alpha = 0, \beta = 0$. Then $\exists C_{2,0}, C_{0,0} \in \mathbb{R}$ such that, for $x \in \mathbb{R}$,

$$\begin{cases} |\varphi(x)| \leq C_{0,0} \\ |x|^2|\varphi(x)| \leq C_{2,0} \end{cases} \quad \Rightarrow \quad |\varphi(x)| \leq \frac{C_{0,0}+C_{2,0}}{1+|x|^2}$$

Hence the claim is proved.

Definizione 2.1.1 *Let X be a vector space. A function $\|\cdot\| : X \longrightarrow \mathbb{R}$ is called a quasi norm if*

(i) $\forall x \in X \ \ \|x\| \geq 0$
(ii) $\forall x \in X$ *if* $\|x\| = 0$ *then* $x = 0$
(iii) $\forall x, j \in X \ \ \|x+y\| \leq \|x\| + \|y\|$
(iv) $\forall x \in X \ \ \|x\| = \|-x\|$

A vector space equipped with a quasi norm enjoying the property of completeness (considering the induced distance $d(x,y) := \|x-y\|$) is called a Frechét space.

Define for each possible choice of $\alpha, \beta \in \mathbb{N} \cup \{0\}$ the function $p_{\alpha,\beta} : \mathscr{S} \longrightarrow \mathbb{R}$

$$p_{\alpha,\beta}(\varphi) := \sup_{x \in \mathbb{R}} |x^\alpha D^\beta \varphi(x)| \tag{2.31}$$

Exercise 2.1.2 *Prove that $p_{\alpha,\beta}$ is a norm, i.e. that for $\varphi, \psi \in \mathscr{S}, \lambda \in \mathbb{R}$*

(i) $p_{\alpha,\beta}(\varphi) \geq 0$
(ii) $p_{\alpha,\beta}(\lambda\varphi) = |\lambda| p_{\alpha,\beta}(\varphi)$
(iii) $p_{\alpha,\beta}(\varphi+\psi) \leq p_{\alpha,\beta}(\varphi) + p_{\alpha,\beta}(\psi)$
(iv) $p_{\alpha,\beta}(\varphi) = 0 \Rightarrow \varphi = 0$

Define the function over $\mathscr{S}$

$$\|\varphi\| := \sum_\alpha \sum_\beta \frac{1}{2^{\alpha+\beta}} \frac{p_{\alpha,\beta}(\varphi)}{1+p_{\alpha,\beta}(\varphi)} \tag{2.32}$$

Exercise 2.1.3 *Prove that $\|\cdot\|$ is a norm on $\mathscr{S}$. Moreover show that $(\mathscr{S}, \|\cdot\|)$ is a Frechét space.*

Theorem 2.1.1 $\varphi \in \mathscr{S} \Rightarrow \widehat{\varphi} \in \mathscr{S}$.

Proof. `First step.` $\varphi \in \mathscr{S}$ `implies that for` $p \in \mathbb{C}[x]$ $p\varphi \in \mathscr{S}$.
Both φ and the identity of $\mathbb{R}$ belong to $C^\infty(\mathbb{R})$, hence the product function $x \mapsto x\varphi(x)$ belongs to $C^{\infty(\mathbb{R})}$. By induction, for $n \in \mathbb{N} \cup \{0\}$,

$$(x\varphi(x))^{(n)} = n\varphi^{(n-1)}(x) + x\varphi^{(n)}(x) \tag{2.33}$$

Indeed, of course the equality holds when $n = 0$. Now, for $n \in \mathbb{N}$, suppose the equality holds when $0 \leq i < n$. Then, noticing that $\varphi^{(1)} \in \mathscr{S}$,

$$
\begin{aligned}
(x\varphi(x))^{(n)} &= \left((x\varphi(x))^{(1)}\right)^{(n-1)} = \left(\varphi(x)+x\varphi^{(1)}(x)\right)^{(n-1)} = \varphi^{(n-1)}(x)+\left(x\varphi^{(1)}(x)\right)^{(n-1)} \\
&= \varphi^{(n-1)}(x)+\left((n-1)(\varphi^{(1)})^{(n-2)}(x)+x(\varphi^{(1)})^{(n-1)}\right) \\
&= n\varphi^{(n-1)}(x)+x\varphi^{(n)}(x)
\end{aligned}
$$

Hence, for $\alpha,\beta\in\mathbb{N}\cup\{0\}$,

$$
|x^{\alpha}(x\varphi(x))^{\beta}| \le |\beta x^{\alpha}\varphi^{(\beta-1)}(x)|+|x^{\alpha+1}\varphi^{(\beta)}(x)| \le \beta C_{\alpha,\beta-1}+C_{\alpha+1,\beta}
$$

implying that $x\in\mathscr{S}$. By a trivial induction, for any $t\in\mathbb{N}$ $x^t\varphi(x)\in\mathscr{S}$, therefore, since $\mathscr{S}$ is a vector space, for $p\in\mathbb{C}[x]$ $p\varphi\in\mathscr{S}$.
`Second step. If` $\varphi\in\mathscr{S}$`, then` $\widehat{\varphi}$ `has derivative over` $\mathbb{R}$ `and` $(\widehat{\varphi})'(\xi)=(-ix\varphi(x))\widehat{}(\xi)$`. Actually,` $\varphi\in C^{\infty}(\mathbb{R})$ `and for` $n\in\mathbb{N}$

$$
(\widehat{\varphi})^{(n)}(\xi)=((-ix)^n\varphi(x))\widehat{}(\xi) \tag{2.34}
$$

Let $\xi,\delta\in\mathbb{R}$. Then

$$
\begin{aligned}
\frac{\widehat{\varphi}(\xi+\delta)-\widehat{\varphi}(\xi)}{(\xi+\delta)-\xi} &= \int_R \frac{e^{-ix(\xi+\delta)}-e^{-ix\xi}}{\delta}\varphi(x)\mathrm{d}x \\
&= \int_R e^{-ix\xi}\frac{e^{-ix\delta}-1}{\delta}\varphi(x)\mathrm{d}x \xrightarrow{\delta\to 0} -i\int_{\mathbb{R}} e^{-ix\xi}x\varphi(x)\mathrm{d}x = (-ix\varphi(x))\widehat{}(\xi)
\end{aligned}
$$

Indeed, $e^{-ix\xi}\frac{e^{-ix\delta}-1}{\delta}\varphi(x) \xrightarrow{\delta\to 0} -ie^{-ix\xi}x\varphi(x)$ and it possible to apply Lebesgue dominated convergence theorem:

$$
\begin{aligned}
\left|\frac{e^{-ix\delta}-1}{\delta}\right| &= \left|\frac{\cos(\delta x)-1-i\sin(\delta x)}{\delta}\right| = \frac{\sqrt{(1-\cos(\delta x))^2+\sin^2(\delta x)}}{|\delta|} \\
&= \frac{\sqrt{2-2\cos(\delta x)}}{|\delta|} = \frac{\sqrt{4\sin^2\left(\frac{x\delta}{2}\right)}}{|\delta|} = \left|\frac{\sin\left(\frac{x\delta}{2}\right)}{\frac{x\delta}{2}}\right|\cdot|x| \le 1\cdot|x| = |x|
\end{aligned}
$$

hence

$$
\left|e^{-ix\xi}\frac{e^{-ix\delta}-1}{\delta}\varphi(x)\right| \le 1\cdot|x|\cdot|\varphi(x)| = |x\varphi(x)|
$$

and $x\mapsto x\varphi(x)$ belongs to $L^1(\mathbb{R})$ since, thanks to the first step, it belongs to $\mathscr{S}$. Hence 2.34 is proved for $n=1$. Now consider $n\ge 2$ and suppose 2.34 holds for $1\le i<n$. Then, thanks to the first step, $(-i)^{n-1}\varphi\in\mathscr{S}$, therefore

$$
\begin{aligned}
((-ix)^n\varphi(x))\widehat{\ }(\xi) &= (-ix\big[(-ix)^{n-1}\varphi(x)\big]\big)\widehat{\ }(\xi) = \Big(\big[(-ix)^{n-1}\varphi(x)\big]\widehat{\ }\Big)^1(\xi)\\
&= \Big(\widehat{\varphi}^{(n-1)}\Big)^1(\xi) = (\widehat{\varphi})^{(n)}(\xi)
\end{aligned}
$$

`Third step. If` $\varphi \in \mathscr{S}$`, then for` $n \in \mathbb{N}$

$$\widehat{\varphi^{(n)}}(\xi) = (i\xi)^n\widehat{\varphi}(\xi) \tag{2.35}$$

For one thing, notice that 2.35 makes sense since if $\varphi \in \mathscr{S}$ then $\varphi^{(n)} \in \mathscr{S}$: clearly, $\varphi^{(n)} \in C^\infty(\mathbb{R})$, and $|x^\alpha D^\beta \varphi^{(n)}(x)| = |x^\alpha D^{\beta+n}\varphi(x)| \le C_{\alpha,\beta+n}$.

$$\widehat{\varphi'}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\varphi'(x)\mathrm{d}x = \lim_{L\to+\infty}\Big[e^{-ix\xi\varphi(x)}\Big]_{-L}^{L} + i\xi\int_{\mathbb{R}} e^{-ix\xi}\varphi(x)\mathrm{d}x = i\xi\widehat{\varphi}(\xi)$$

And this proves 2.35 for $n = 1$. Take $n \le 2$ and, by induction, suppose 2.35 holds for $1 \le j < n$. Then

$$\widehat{\varphi^{(n)}}(\xi) = (\widehat{\varphi^{(1)})^{(n-1)}}(\xi) = (i\xi)^{n-1}\widehat{\varphi^{(1)}}(\xi) = (i\xi)^{n-1}(i\xi)^1\widehat{\varphi}(\xi) = (i\xi)^n\widehat{\varphi}(\xi)$$

`Fourth step.` $\varphi \in \mathscr{S} \Rightarrow \forall\xi \in \mathbb{R}\ |\widehat{\varphi}(\xi)| \le \|\varphi\|_{L^1} \in \mathbb{R}$.

$$|\widehat{\varphi}(\xi)| = \left|\int_{\mathbb{R}} e^{-ix\xi}\varphi(x)\mathrm{d}x\right| \le \int_{\mathbb{R}} |\varphi(x)|\mathrm{d}x = \|\varphi\|_{L^1} \tag{2.36}$$

`Fifth step. Conclusion:` $\alpha,\beta \in \mathbb{N}\cup\{0\}$ $\exists E_{\alpha,\beta} \in \mathbb{R}$ `such that` $|x^\alpha D^\beta \widehat{\varphi}(x)| \le E_{\alpha,\beta}$.
Using the previous steps,

$$
\begin{aligned}
|\xi^\alpha(\widehat{\varphi})^{(\beta)}(\xi)| &\overset{2.34}{=} |\xi^\alpha\big((-ix)^\beta\varphi(x)\big)\widehat{\ }(\xi)| = |(i\xi)^\alpha[(-i)^{\alpha+\beta}x^\beta\varphi(x)]\widehat{\ }(\xi)|\\
&\overset{2.35}{=} |\Big([(-i)^{\alpha+\beta}x^\beta\varphi(x)]^{(\alpha)}\Big)\widehat{\ }(\xi)| \overset{2.36}{\le} \|[(-i)^{\alpha+\beta}x^\beta\varphi(x)]^{(\alpha)}\|_{L^1} \in \mathbb{R}
\end{aligned}
$$

Hence the claim is proved defining $E_{\alpha,\beta} := \|[(-i)^{\alpha+\beta}x^\beta\varphi(x)]^{(\alpha)}\|_{L^1}$.

Define $R : \mathscr{S} \longrightarrow \mathscr{S}$ as, for $\xi \in \mathbb{R}$, $\big(R\varphi\big)(\xi) := \varphi(-\xi)$. Easily, the codomain of R is $\mathscr{S}$ since $R\varphi \in C^\infty(\mathbb{R})$ and $|\xi^\alpha D^{(\beta)}\big(R\varphi\big)(\xi)| = |\xi^\alpha(-1)^\beta\big(D^{(\beta)}\varphi\big)(-\xi)| \overset{y\overset{s}{=}-\xi}{=} |y^\alpha D^\beta(y)| \le C_{\alpha,\beta}$. Then, thanks to the preceding theorem, one gets the following commutative diagram

$$
\begin{array}{ccc}
\mathscr{S} & \xrightarrow{\ \widehat{\ }\ } & \mathscr{S}\\
\Big\downarrow R & \circlearrowleft & \Big\downarrow R\\
\mathscr{S} & \xrightarrow{\ \widehat{\ }\ } & \mathscr{S}
\end{array}
$$

Indeed, the diagram is commutative since, for $\varphi \in \mathscr{S}$

$$\big(R\widehat{\varphi}\big)(\xi) = \widehat{\varphi}(-\xi) = \int_{-\infty}^{+\infty} e^{i\xi x}\varphi(x)\mathrm{d}x \overset{y\overset{s}{=}-x}{=} -\int_{+\infty}^{-\infty} e^{-i\xi y}\varphi(-y)\mathrm{d}y$$
$$= \int_R e^{-i\xi y}\big(R\varphi\big)(y)\mathrm{d}y = \widehat{R\varphi}(\xi)$$

Since $\check{\varphi}(\xi) = \frac{1}{2\pi}\big(R\varphi\big)(\xi)$, thanks to the homogeneity of the integral and to the fact that $\mathscr{S}$ is a vector space, one gets the commutative diagram

$$\begin{array}{ccc} \mathscr{S} & \xrightarrow{\ \hat{}\ } & \mathscr{S} \\ \big\downarrow \check{} & \circlearrowleft & \big\downarrow \check{} \\ \mathscr{S} & \xrightarrow{\ \hat{}\ } & \mathscr{S} \end{array}$$

which includes the theorem

Theorem 2.1.2 $\varphi \in \mathscr{S} \Rightarrow \check{\varphi} \in \mathscr{S}$.

As a consequence of the two last theorems

Corollary 2.1.1 *If* $\varphi \in \mathscr{S}$*, then* $\check{\hat{\varphi}} = \varphi = \hat{\check{\varphi}}$.

Notice that the foregoing theorem is a special case of the third step of Example 2.0.3: if $\varphi \in \mathscr{S}$, then $\varphi \in L^1(\mathbb{R}) \cap C_b(\mathbb{R})$ and $\widehat{\varphi} \in L^1(\mathbb{R})$.

2.2 Fourier transform of functions in $L^2(\mathbb{R})$

Exercise 2.2.1 *Show that*

1. $\varphi, \psi \in \mathscr{S} \Rightarrow \varphi\psi \in \mathscr{S}$
2. $\forall p \in [1, +\infty) \quad \mathscr{S} \subset L^p(\mathbb{R})$

Solution of Exercise 2.2.1. Obviously $\varphi\psi \in C^\infty(\mathbb{R})$ and moreover, for $n \in \mathbb{N}$

$$\big(\varphi\psi\big)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \varphi^{(k)}\psi^{(n-k)} \tag{2.37}$$

Indeed, for $n = 1$ $\big(\varphi\psi\big)^{(1)} = \binom{1}{0}\varphi^{(0)}\psi^{(1)} + \binom{1}{1}\varphi^{(1)}\psi^{(0)}$, and for $n \geq 2$, assuming the formula holds for $1 \leq i \leq n-1$

$$\begin{aligned}
(\varphi\psi)^{(n)} &= \left((\varphi\psi)^{(n-1)}\right)^{(1)} = \left(\sum_{k=0}^{n-1}\binom{n-1}{k}\varphi^{(k)}\psi^{(n-1-k)}\right)^{(1)} \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}\left[\varphi^{(k+1)}\psi^{(n-(k+1))}+\varphi^{(k)}\psi^{(n-k)}\right] \\
&= \varphi^{(0)}\psi^{(n)} + \sum_{k=1}^{n-1}\left[\binom{n-1}{k-1}+\binom{n-1}{k}\right]\varphi^{(k)}\psi^{(n-k)} + \varphi^{(n)}\psi^{(0)} \\
&= \sum_{k=0}^{n}\binom{n}{k}\varphi^{(k)}\psi^{(n-k)}
\end{aligned}$$

Hence, for $\alpha, \beta \in \mathbb{N}\cup\{0\}$,

$$|x^\alpha D^\beta(\varphi\psi)(x)| \le \sum_{k=0}^{\beta}\binom{\beta}{k}|x^\alpha\varphi^{(k)}(x)\psi^{(\beta-k)}(x)| \le \sum_{k=0}^{\beta}\binom{\beta}{k}C^{\varphi}_{\alpha,k}C^{\psi}_{0,\beta-k} \in \mathbb{R}$$

and the first point of the exercise is proved. For the second point notice that thanks to the first, by trivial induction, $\forall p \in \mathbb{N}$ if $\varphi \in \mathscr{S}$ then $\varphi^p \in \mathscr{S}$. Therefore for $p \in [1, +\infty)$

$$\begin{aligned}
\int_{\mathbb{R}}|\varphi|^p(x)\mathrm{d}x &= \int_{|\varphi(x)|<1}|\varphi|^p(x)\mathrm{d}x + \int_{|\varphi(x)|\ge 1}|\varphi^p|(x)\mathrm{d}x \\
&\le \int_{|\varphi(x)|<1}|\varphi|^{\lfloor p\rfloor}(x)\mathrm{d}x + \int_{|\varphi(x)|\ge 1}|\varphi|^{\lfloor p\rfloor+1}(x)\mathrm{d}x \\
&\le \int_{\mathbb{R}}|\varphi^{\lfloor p\rfloor}|(x)\mathrm{d}x + \int_{\mathbb{R}}|\varphi^{\lfloor p\rfloor+1}|(x)\mathrm{d}x \overset{\mathscr{S}\subset L^1(\mathbb{R})}{=} \|\varphi^{\lfloor p\rfloor}\|_{L^1} + \|\varphi^{\lfloor p\rfloor+1}\|_{L^1}
\end{aligned}$$

and this means $\varphi^p \in L^p(\mathbb{R})$.

In particular, $\mathscr{S} \subset L^2(\mathbb{R})$.

Theorem 2.2.1 *Let $\varphi, \psi \in \mathscr{S}$. Then*

$$\langle\varphi,\psi\rangle = \frac{1}{2\pi}\langle\hat{\varphi},\hat{\psi}\rangle \tag{2.38}$$

$$\langle\check{\varphi},\check{\psi}\rangle = \frac{1}{2\pi}\langle\varphi,\psi\rangle \tag{2.39}$$

Proof. First, notice that $\psi \in \mathscr{S} \quad\Rightarrow\quad \overline{\psi} \in \mathscr{S}$. Moreover

$$\check{\overline{\psi}}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\overline{\psi}(x)\mathrm{d}x = \frac{1}{2\pi}\overline{\int_{\mathbb{R}} e^{-i\xi x}\psi(x)\mathrm{d}x} = \frac{1}{2\pi}\overline{\hat{\psi}}(\xi)$$

Then, since the Fourier transform and the inverse Fourier transform are bijective over $\mathscr{S}$,

$$\langle \varphi, \psi \rangle = \int_{\mathbb{R}} \varphi(x)\overline{\psi}(x)\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\hat{\check{\overline{\psi}}}(x)\mathrm{d}x \overset{Proposition\ 2.0.1}{=} \int_{\mathbb{R}} \hat{\varphi}(x)\check{\overline{\psi}}(x)\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\varphi}(x)\overline{\hat{\psi}}(x)\mathrm{d}x = \frac{1}{2\pi}\langle \widehat{\varphi}, \widehat{\psi} \rangle$$

And since $\check{\varphi}, \check{\psi} \in \mathscr{S}$

$$\langle \check{\varphi}, \check{\psi} \rangle = \frac{1}{2\pi}\langle \hat{\check{\varphi}}, \hat{\check{\psi}} \rangle = \frac{1}{2\pi}\langle \varphi, \psi \rangle$$

Remark 7 *$C_c^\infty(\mathbb{R}) \subset \mathscr{S}$, where $C_c^\infty(\mathbb{R})$ denotes the set of functions in C^∞ having compact support.*

Since $C_c^\infty(\mathbb{R}) \subset \mathscr{S} \subset L^2(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ is dense in $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$, $\mathscr{S}$ is dense in $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$. Hence, if $f \in L^2(\mathbb{R})$ there exists a $\mathscr{S}$-valued sequence $\{\varphi_n\}_n$ that converges to f with respect to the L^2-norm. One can consider the sequence $\{\hat{\varphi}_n\}_n$ and

$$\|\varphi_n - \varphi_m\|_{L^2}^2 \overset{Theorem\ 2.2.1}{=} \frac{1}{2\pi}\|\widehat{\varphi_n - \varphi_m}\|_{L^2}^2 = \frac{1}{2\pi}\|\widehat{\varphi_n} - \widehat{\varphi_m}\|_{L^2}^2$$

Hence $\{\hat{\varphi}_n\}_n$ is a Cauchy sequence and since $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$ is a Banach space $\exists g \in L^2(\mathbb{R})$ such that $\widehat{\varphi_n} \xrightarrow{L^2} g$.

Definition 2.2.1 *Let $f \in L^2(\mathbb{R})$. Then define the Fourier transform $\hat{f}$ of f as $\hat{f} := g$ where $g \in L^2(\mathbb{R})$ is the function constructed above. The inverse Fourier transform of f, $\check{f}$, is defined, by an analogous construction, as the limit in L^2 of the sequence $\{\check{\varphi}_n\}_n$.*

Moreover, from Theorem 2.2.1 and the L^2-convergence one gets that for $f, g \in L^2(\mathbb{R})$

$$\langle f, g \rangle = \frac{1}{2\pi}\langle \hat{f}, \hat{g} \rangle \tag{2.40}$$

$$\langle \check{f}, \check{g} \rangle = \frac{1}{2\pi}\langle f, g \rangle \tag{2.41}$$

In particular

$$\|f\|_{L^2}^2 = \frac{1}{2\pi}\|\hat{f}\|_{L^2}^2 \tag{2.42}$$

$$\|\check{f}\|_{L^2}^2 = \frac{1}{2\pi}\|f\|_{L^2}^2 \tag{2.43}$$

Exercise 2.2.2 *The Fourier transform and the inverse Fourier transform over $L^2(\mathbb{R})$ are well-defined, i.e. they do not depend on the chosen converging sequence $\{\varphi_n\}_n$.*

The Fourier transform and the inverse Fourier transform over $L^2(\mathbb{R})$ are L^2-valued because of the completeness of $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$. This in particular implies that they can be applied in a sequence to a function. From the definition, chosing appropriately the converging sequences, one gets that the commutativity of the following diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\;\hat{}\;} & L^2(\mathbb{R}) \\ \downarrow \check{} & \circlearrowleft & \downarrow \check{} \\ L^2(\mathbb{R}) & \xrightarrow{\;\hat{}\;} & L^2(\mathbb{R}) \end{array}$$

and for $f \in L^2(\mathbb{R})$ $\check{\hat{f}} = f = \hat{\check{f}}$.

2.3 An application to the heat equation on a line

Consider the problem (that can be seen as the equation modelling the diffusion of heat along an infinitely long bar, with t regarded as the time ccordinate and x as the position)

$$\begin{cases} u_t = cu_{xx} & \text{in } \{(t,x) \in \mathbb{R}^2 : t > 0\} \\ u(0,x)) = f(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{2.44}$$

where the unknown function $u(t,x)$ is defined over $\mathbb{R}^+ \times \mathbb{R}$, $u(t,\cdot) \in L^2(\mathbb{R})$ (but notice that one has to derive $u(t,\cdot)$) and $f \in L^2(\mathbb{R})$, $c \in \mathbb{R}$, $c > 0$ are given.
Actually, only the uniqueness of the solution will be proved here. In order to achieve this goal, suppose that $u(t,\cdot)$ admits a Fourier transform as functions in $L^1(\mathbb{R})$ and that the transform is invertible. Then

$$\int_{\mathbb{R}} \frac{\partial u}{\partial t}(t,x) e^{-i\xi x} \mathrm{d}x = \frac{\partial}{\partial t} \int_{\mathbb{R}} u(t,x) e^{-i\xi x} \mathrm{d}x = \frac{\partial}{\partial t} \hat{u}(t,\xi)$$

Then, recalling 2.35 (that actually holds for functions in $\mathscr{S}$),

$$\int_{\mathbb{R}} \frac{\partial^2 u}{\partial x^2}(t,x) e^{-i\xi x} \mathrm{d}x = -\xi^2 \hat{u}(t,\xi)$$

Therefore the original problem becomes

$$\begin{cases} (\hat{u})_t = -c\xi^2 \hat{u} & \text{in } \{(t,x) \in \mathbb{R}^2 : t > 0\} \\ \hat{u}(0,x)) = \hat{f}(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{2.45}$$

The solution of the ODE is

$$\hat{u}(t,\xi) = \hat{f}(\xi) e^{-c\xi^2 t} = \hat{f}(\xi) \left(\frac{1}{\sqrt{4\pi ct}} e^{-\frac{x^2}{4ct}} \right)\hat{}(\xi) \tag{2.46}$$

Remark 8 *If $f,g \in L^1(\mathbb{R})$ then for $\xi \in \mathbb{R}$*

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi) \tag{2.47}$$

Indeed,

$$\widehat{(f*g)}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} \left[\int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y \right] \mathrm{d}x = \int_{\mathbb{R}} g(y) \left[\int_{\mathbb{R}} f(x-y)e^{-ix\xi}\mathrm{d}x \right] \mathrm{d}y$$

$$= \int_{\mathbb{R}} g(y)e^{-iy\xi} \left[\int_{\mathbb{R}} f(x-y)e^{-i(x-y)\xi}\mathrm{d}x \right] \mathrm{d}y = \int_{\mathbb{R}} g(y)e^{-iy\xi}\hat{f}(\xi)\mathrm{d}y = \hat{f}(\xi)\hat{g}(\xi)$$

Hence

$$\hat{u}(t,\xi) = \frac{1}{\sqrt{4\pi ct}} \left(\widehat{f * e^{-\frac{x^2}{4ct}}} \right) \tag{2.48}$$

And if the Fourier transform is invertible

$$u(t,\xi) = \frac{1}{\sqrt{4\pi ct}} \left(f * e^{-\frac{x^2}{4ct}} \right) = \frac{1}{\sqrt{4\pi ct}} \int_{\mathbb{R}} f(y)e^{-\frac{(x-y)^2}{4ct}}\mathrm{d}y \tag{2.49}$$

The problem in 2.44 belongs to the family of problems called Cauchy problems or evolution equations. The next problem belongs to the same family. The Schrödinger equation (quantum free particle) is

$$\begin{cases} u_t = icu_{xx} & \text{in } \{(t,x) \in \mathbb{R}^2 : t > 0\} \\ u(0,x) = f(x) & \text{for } x \in \mathbb{R} \end{cases} \tag{2.50}$$

where $f \in L^1(\mathbb{R})$, $c \in \mathbb{R}$ and $u(t,\cdot) \in L^1(\mathbb{R})$. Again, applying the Fourier transform to both sides, one gets

$$\begin{cases} \hat{u}_t = ic\xi\hat{u} \\ \hat{u}(0,x) = \hat{f}(x) \end{cases} \tag{2.51}$$

hence the solution of the ODE is $\hat{u}(t,\xi) = e^{ic\xi t}\hat{f}(\xi)$, but this time for every $p \in [1,+\infty)$ $e^{ic\xi t} \notin L^p(\mathbb{R})$ since

$$\int_{\mathbb{R}} |e^{ic\xi t}|^p \mathrm{d}\xi = \int_{\mathbb{R}} 1\mathrm{d}x = +\infty \tag{2.52}$$

and this implies that one cannot find the solution applying the inverse Fourier transform, in one of the versions given before, to $\xi \mapsto e^{ic\xi t}$. However, define $v(t,x) := f(x+ct)$, then, since $f \in L^1(\mathbb{R})$,

$$\hat{v}(t,\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x+ct)\mathrm{d}x = \int_{\mathbb{R}} e^{-i\xi(x-ct)} f(x)\mathrm{d}x = e^{\xi ct} \int_{\mathbb{R}} e^{-ix\xi} f(x)\mathrm{d}x = e^{ic\xi t}\hat{f}(\xi)$$

and it follows that $u(t,x) = f(x+ct)$.

2.4 Extension of the Fourier transform

If $(\mathscr{X}, \|\cdot\|)$ is a normed vector space (over a field K, that might be $\mathbb{C}$ or $\mathbb{R}$), its dual is the set

$$\mathscr{X}' := \Big\{ \varphi \in \mathbb{C}^{\mathscr{X}} : \quad \forall x, y \in \mathscr{X}, \lambda \in K \quad \varphi(x + \lambda y) = \varphi(x) + \lambda \varphi(y) \\ \text{and} \quad \exists c \in \mathbb{R} \, \forall x \in \mathscr{X} \ |\varphi(x)| \le c\|x\| \Big\}$$

and defining $\|\cdot\|_{\mathscr{X}'} : \mathscr{X}^* \longrightarrow \mathbb{R}$ as

$$\|\varphi\|_{\mathscr{X}'} := \sup_{x \neq 0} \frac{|\varphi(x)|}{\|x\|} = \sup_{\|x\| \le 1} |\varphi(x)| \tag{2.53}$$

one has that $(\mathscr{X}', \|\cdot\|_{\mathscr{X}'})$ is a Banach space. Actually, an other definition of dual is sometimes applied: the algebraic dual of $\mathscr{X}$, $\mathscr{X}^*$, is the space of all the functionals (linear functions) from the vector space $\mathscr{X}$ to its field K, i.e. the functionals are not continuous in general. Indeed the latter definition makes sense in vector spaces that are not topological spaces. Obviously $\mathscr{X}' \subset \mathscr{X}^*$.
For $p \in \mathbb{R}$, $1 < p < +\infty$, $(L^p(\mathbb{R}), \|\cdot\|_{L^p})$ is a Banach space and it the dual $\Big(L^p(\mathbb{R})\Big)'$ is identified with $L^q(\mathbb{R})$ where q is such that $\frac{1}{p} + \frac{1}{q} = 1$ via the isomorphism $\Big(L^p(\mathbb{R})\Big)' \longrightarrow L^q(\mathbb{R})$ that maps $\varphi \in \Big(L^p(\mathbb{R})\Big)'$ to the function $g \in L^q(\mathbb{R})$, given by Riesz representation theorem, such that

$$\forall p \in L^p(\mathbb{R}) \qquad \varphi(f) = \int_{\mathbb{R}} f(x) g(x) \mathrm{d}x \tag{2.54}$$

For short, one says that the dual of $L^p(\mathbb{R})$ is $L^q(\mathbb{R})$. A particular case is the one of $p = 2$ since $p = q = 2$. Actually, $(L^2(\mathbb{R}), \|\cdot\|_{L^2})$ is an Hilbert space, and more generally, if H is a set endowed with a structure of Hilbert space, then, for short, $H' = H$.
The vector space $\mathscr{S}$ is a topological space. A normed vector space can be endowed with the topology induced by the norm, but a topology is not always induced by a norm. Hence in the case of $\mathscr{S}$ the dual can be defined as

$$\mathscr{S}' := \Big\{ T \in \mathbb{C}^{\mathscr{S}} : \forall \varphi, \psi \in \mathscr{S}, \lambda \in \mathbb{C} \quad T(\varphi + \lambda \psi) = T(\varphi) + \lambda T(\psi) \\ \text{and if} \quad \varphi_n \xrightarrow{\mathscr{S}} \varphi \text{ then } T(\varphi_n) \to T(\varphi) \Big\} \tag{2.55}$$

$\mathscr{S}'$ is a vector space, but not a normed one. It is called the vector space of tempered disributions ($T \in \mathscr{S}'$ is called a tempered distribution) or it is also called the space of generalized functions.

Definition 2.4.1 *Let $T \in \mathscr{S}'$. Then the generalized derivative (or the derivative in the sense of the theory of distributions, or derivative in the weak sense) is, for $\varphi \in$*

$\mathscr{S}$,

$$T'(\varphi) := -T(\varphi') \tag{2.56}$$

The Fourier transform and the inverse Fourier transform are, for $\varphi \in \mathscr{S}$,

$$\hat{T}(\varphi) := T(\hat{\varphi}) \tag{2.57}$$

$$\check{T}(\varphi) := T(\check{\varphi}) \tag{2.58}$$

Notice that the definitions are good definitions since if $\varphi \in \mathscr{S}$ then $\varphi', \hat{\varphi}, \check{\varphi} \in \mathscr{S}$. One has that $T', \hat{T}, \check{T} \in \mathscr{S}'$. Moreover

$$\check{\hat{T}}(\varphi) = \hat{T}(\check{\varphi}) = T(\hat{\check{\varphi}}) = T(\varphi) \tag{2.59}$$

that is to say, $\check{\hat{T}} = T$ (and also $\hat{\check{T}} = T$).
$\mathscr{S}$ can be embedded in $\mathscr{S}'$: for $\psi, \varphi \in \mathscr{S}$

$$T_\psi(\varphi) := \int_{\mathbb{R}} \varphi(x)\psi(x)\mathrm{d}x \tag{2.60}$$

$T_\psi \in \mathscr{S}'$, and so $\mathscr{S} \subset \mathscr{S}'$ makes sense via the identification of ψ and T_ψ. Now,

$$T'_\psi(\varphi) = -T_\psi(\varphi') = -\int_{\mathbb{R}} \varphi'(x)\psi(x)\mathrm{d}x = -\lim_{L\to+\infty}\Big[\varphi(x)\psi(x)\Big]_{-L}^{L} + \int_{\mathbb{R}} \varphi(x)\psi'(x)\mathrm{d}x$$
$$\overset{\varphi\psi\in\mathscr{S}}{=} \int_{\mathbb{R}} \varphi(x)\psi'(x)\mathrm{d}x = T_{\psi'}(\varphi)$$

$$\hat{T}_\psi(\varphi) = T_\psi(\hat{\varphi}) = \int_{\mathbb{R}} \hat{\varphi}(x)\psi(x)\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\hat{\psi}(x)\mathrm{d}x = T_{\hat{\psi}}(\varphi)$$

$$\check{T}_\psi(\varphi) = T_\psi(\check{\varphi}) = \int_{\mathbb{R}} \check{\varphi}(x)\psi(x)\mathrm{d}x = \int_{\mathbb{R}} \varphi(x)\check{\psi}(x)\mathrm{d}x = T_{\check{\psi}}(\varphi)$$

that is, $\psi', \hat{\psi}$ and $\check{\psi}$ are identified with $T', \hat{\psi}$ and $\check{\psi}$ respectively. This explains the names given to $T', \hat{\psi}$ and $\check{\psi}$. Actually, a set of functions larger than $\mathscr{S}$ can be embedded in $\mathscr{S}'$ via an extension of 2.60. For instance, consider the Heaviside function $H : \mathbb{R} \longrightarrow \mathbb{C}$

$$H(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{2.61}$$

Then, for $\varphi \in \mathscr{S}$,

$$H \longrightarrow T_H(\varphi) = \int_{\mathbb{R}} \varphi(x)H(x)\mathrm{d}x = \int_0^{+\infty} \varphi(x)\mathrm{d}x \tag{2.62}$$

$T_H \in \mathscr{S}'$ and

$$T'_H(\varphi) = -T_H(\varphi') = -\int_0^{+\infty} \varphi'(x)\mathrm{d}x = \varphi(0) = \int_{\mathbb{R}} \varphi(x)\delta_0(\mathrm{d}x)$$

i.e. the derivative in the weak sense of the Heaviside function is the measure Delta of Dirac in 0, δ_0: $\mathscr{S}'$ also "contains "measures, it does not contain just functions. $\mathscr{S}'$ contains elements that are neither functions nor measures (i.e. that do not come from a function or a measure over $\mathbb{R}$ via the extension of 2.60), for instance

$$T'_{\delta_0}(\varphi) = T_{\delta_0}(\varphi') = \varphi'(0)$$

i.e., $\varphi \mapsto \varphi'(0)$. Carrying on this example,

$$\hat{T}_{\delta_0}(\varphi) = T_{\delta_0}(\hat{\varphi}) = \hat{\varphi}(0) = \int_{\mathbb{R}} \varphi(x)\mathrm{d}x = \int_{\mathbb{R}} 1 \cdot \varphi(x)\mathrm{d}x = T_1(\varphi)$$
$$\check{T}_1(\varphi) = \hat{T}_{\delta_0}(\check{\varphi}) = T_{\delta_0}(\hat{\check{\varphi}}) = T_{\delta_0}(\varphi)$$

thus $\hat{T}_{\delta_0} = T_1$, and often the two above results are written as $\hat{\delta}_0 = 1$ and $\check{1} = \check{\hat{\delta}} = \delta$. Other spaces that can be embedded in $\mathscr{S}'$ are $L^p(\mathbb{R})$, $C_0(\mathbb{R})$, $C_b(\mathbb{R})$ and $\mathbb{C}[x]$. If $p \in \mathbb{C}[x]$ then it has already been shown that $p\varphi \in \mathscr{S}$ for any $\varphi \in \mathscr{S}$, and therefore it is possible to compute $T_p(\varphi) = \int_{\mathbb{R}} \varphi(x)p(x)\mathrm{d}x$.
Not the whole $C(\mathbb{R})$, space of continuous functions, can be embedded in $\mathscr{S}'$. Indeed, if for $f \in C(\mathbb{R})$ $\exists A \in \mathbb{R}, m \in \mathbb{N}$ such that for $x \in \mathbb{R}$

$$|f(x)| \leq A + |x|^m \tag{2.63}$$

then f can be embedded in $\mathscr{S}$. For instance the functions e^x and e^{-x} cannot be embedded in $\mathscr{S}$ (loosely speaking, they can be seen as infinite polynomials and their growth is too quick to be dampened by that of $\varphi \in \mathscr{S}$). Actually, $\mathscr{S}'$ can be embedded in a larger space, $\mathscr{D}'(\mathbb{R})$, that contains also e^x and e^{-x}. $\mathscr{D}'(\mathbb{R})$ is constructed as the dual space of $\mathscr{D}(\mathbb{R})$, that is the set C_c^∞ endowed with a particular topology.

2.5 Fourier transform in $\mathbb{R}^d$, for $d \in \mathbb{N}$

Definition 2.5.1 *Consider $f \in L^1(\mathbb{R}^d)$, $f : \mathbb{R}^d \to \mathbb{C}$. The Fourier transform of f is the function $\hat{f} : \mathbb{R}^d \to \mathbb{C}$ defined, for $\xi \in \mathbb{R}^d$, as*

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\langle x,\xi\rangle} f(x)\mathrm{d}x \tag{2.64}$$

The inverse Fourier transform of f is the function $\check{f} : \mathbb{R}^d \to \mathbb{C}$ defined, for $\xi \in \mathbb{R}^d$, as

$$\check{f}(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\langle x,\xi\rangle} f(x)\mathrm{d}x \tag{2.65}$$

As in the case $d = 1$, $\hat{f}, \check{f} \in C_0(\mathbb{R}^d)$, moreover

Theorem 2.5.1 *Let $f, g \in L^1(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} f(x)\hat{g}(x)\mathrm{d}x = \int_{\mathbb{R}^d} \hat{f}(x)g(x)\mathrm{d}x \tag{2.66}$$

Define

$$\mathscr{S}(\mathbb{R}^d) := \Big\{ \varphi \in C^\infty(\mathbb{R}^d) :\ \forall m = (m_1,...,m_d), n = (n_1,...,n_d) \in \Big(\mathbb{N}\cup\{0\}\Big)^d \\ \exists C_{m,n} \in \mathbb{R} \text{ s.t. } \forall \xi = (\xi_1,...,\xi_d) \in \mathbb{R}^d \quad \Big| \prod_{i=1}^d \xi_i^{m_i} \frac{\partial^{\Sigma_{i=1}^d n_i}\varphi}{\partial x_1^{n_1}\cdots\partial x_d^{n_d}}(\xi) \Big| \le C_{m,n} \Big\} \tag{2.67}$$

$\mathscr{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$ is a vector space.

Theorem 2.5.2 *If $\varphi \in \mathscr{S}(\mathbb{R}^d)$, then $\hat{\varphi}, \check{\varphi} \in \mathscr{S}(\mathbb{R}^d)$. Moreover $\hat{\check{\varphi}} = \varphi = \check{\hat{\varphi}}$, and again, as for $d = 1$,*

$$\begin{array}{ccc} \mathscr{S}(\mathbb{R}^d) & \xrightarrow{\ \hat{}\ } & \mathscr{S}(\mathbb{R}^d) \\ \big\downarrow \check{} & \circlearrowleft & \big\downarrow \check{} \\ \mathscr{S}(\mathbb{R}^d) & \xrightarrow{\ \hat{}\ } & \mathscr{S}(\mathbb{R}^d) \end{array}$$

Exactly as for $d = 1$, by means of approximation, one can extend the Fourier transform to functions $f \in L^2(\mathbb{R}^d)$. And as before, for $f, g \in L^2(\mathbb{R}^d)$

$$\langle f, g\rangle = \frac{1}{(2\pi)^d}\langle \hat{f}, \hat{g}\rangle \tag{2.68}$$

Define, for $m = (m_1,...,m_d), n = (n_1,...,n_d) \in \Big(\mathbb{N}\cup\{0\}\Big)^d$, $p_{m,n} : \mathscr{S}(\mathbb{R}^d) \longrightarrow \mathbb{R}$

$$p_{m,n}(\varphi) := \sup_{\xi\in\mathbb{R}^d} \Big| \prod_{i=1}^d \xi_i^{m_i} \frac{\partial^{\Sigma_{i=1}^d n_i}\varphi}{\partial x_1^{n_1}\cdots\partial x_d^{n_d}}(\xi) \Big| \tag{2.69}$$

and thus, again, $\mathscr{S}(\mathbb{R}^d)$ can be endowed with a quasi norm that turns $\mathscr{S}(\mathbb{R}^d)$ into a Frechét space. The dual of $\mathscr{S}(\mathbb{R}^d)$ is

$$\mathscr{S}'(\mathbb{R}^d) := \Big\{ T \in \mathbb{C}^{\mathscr{S}(\mathbb{R}^d)} :\ T \text{ is a continuous linear function} \Big\} \tag{2.70}$$

One can define the Fourier transform and the inverse Fourier transform over $\mathscr{S}'(\mathbb{R}^d)$ as $\hat{T}(\varphi) := T(\hat{\varphi})$ and $\check{T}(\varphi) := T(\check{\varphi})$ getting the commutative diagram

$$\begin{array}{ccc} \mathscr{S}'(\mathbb{R}^d) & \xrightarrow{\ \hat{}\ } & \mathscr{S}'(\mathbb{R}^d) \\ \big\downarrow \check{} & \circlearrowleft & \big\downarrow \check{} \\ \mathscr{S}'(\mathbb{R}^d) & \xrightarrow{\ \hat{}\ } & \mathscr{S}'(\mathbb{R}^d) \end{array}$$

2.6 An application to PDE

Consider the equation

$$\lambda u - \Delta u = f \tag{2.71}$$

where $\lambda \in \mathbb{C}$, Δ is the laplacian and $u, f : \mathbb{R}^d \longrightarrow \mathbb{C}$. f is given, but for the time being no further hypotheses concerning the spaces where f and u belong are stated. One can apply to both the members of the equation the Fourier transform (not specifying what Fourier transform and using the properties that all the definitions share)

$$\begin{aligned}\hat{f}(\xi) &= (\lambda u - \Delta u)\hat{}(\xi) = \lambda \hat{u}(\xi) - (\Delta u)\hat{}(\xi) \\ &= \lambda \hat{u}(\xi) - \sum_{k=1}^{d} \Big(\frac{\partial^2 u}{\partial x_k^2}\Big)\hat{}(\xi) = \lambda \hat{u}(\xi) - \sum_{k=1}^{d} (i\xi_k)^2 \hat{u}(\xi) \\ &= \lambda \hat{u}(\xi) + \hat{u} \sum_{k=1}^{d} \xi_k^2 = (\lambda + |\xi|^2)\hat{u}(\xi) \end{aligned} \tag{2.72}$$

Now suppose $f \in L^2(\mathbb{R}^d)$. Then $\hat{f} \in L^2(\mathbb{R}^d)$ and if $\lambda \notin \{x \in \mathbb{R} : x \leq 0\}$ then $\frac{1}{\lambda + |\xi|^2} \in C_0$, hence $\frac{\hat{f}}{\lambda + |\xi|^2} \in L^2(\mathbb{R}^d)$ and

$$u(\xi) = \left(\frac{\hat{f}}{\lambda + |\cdot|^2}\right)\check{}(\xi) \in L^2(\mathbb{R}^d) \tag{2.73}$$

In this case

$$\begin{aligned}\|u\|_{L^2}^2 = \frac{1}{(2\pi)^d}\|\hat{u}\|_{L^2}^2 = \frac{1}{(2\pi)^d}\left\|\frac{\hat{f}}{\lambda + |\cdot|^2}\right\|_{L^2}^2 &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{|\lambda + |\xi|^2|^2} \mathrm{d}\xi \\ &\leq \frac{1}{(2\pi)^2}\frac{1}{|\lambda|^2}\|\hat{f}\|_{L^2}^2 = \frac{1}{|\lambda|^2}\|f\|_{L^2}^2\end{aligned}$$

A more general version of 2.71 is

$$\lambda u - \Big(\Delta u + \langle \beta, \nabla u \rangle\Big) = f \tag{2.74}$$

where $\lambda \in \mathbb{C}$ and $\beta \in \mathbb{R}^d$. For $\beta = 0$ this equation is exactly the previous one. Applying again the Fourier transform, (notice that $\frac{\partial \hat{u}}{\partial x_k}(\xi) = i\xi_k \hat{u}$)

$$\hat{f} = \lambda \hat{u} - \Big(-|\xi|^2 \hat{u} + i\langle \beta, \xi \rangle \hat{u}\Big) = \Big(\lambda + |\xi|^2 - i\langle \beta, \xi \rangle\Big)\hat{u}$$

And if $\forall \xi \in \mathbb{R}^d$ $\lambda + |\xi|^2 - i\langle \beta, \xi \rangle \neq 0$ then

$$u(\xi) = \left(\frac{\hat{f}}{\lambda + |\cdot|^2 - i\langle \beta, \xi \rangle}\right)\check{}(\xi) \in L^2(\mathbb{R}^d) \tag{2.75}$$

A futher generalization is possible: if $A = (a_{kh})$ is a $d \times d$ positive definite matrix and $d \in \mathbb{R}^d$, then one might look for the solutions of

$$\lambda u - \left(\sum_{k,h=1}^{d} a_{kh} \frac{\partial^2 u}{\partial x_k \partial x_h} + \langle \beta, \nabla u \rangle \right) = f \tag{2.76}$$

Chapter 3
Laplace Transform

The space $L^1_{loc}(\mathbb{R}^+)$ is the spaces of all the functions f defined over $\mathbb{R}^+$ such that for any $T \in \mathbb{R}^+$ $f \in L^1(0,T)$.
Suppose that for $f \in L^1_{loc}(\mathbb{R}^+)$ there exist $M, c, \alpha \in \mathbb{R}^+$ such that

$$\forall t \in \mathbb{R}, t \geq c \qquad |f(t)| \leq Me^{\alpha t} \tag{3.1}$$

Then for $p \in \mathbb{C}$ such that $\mathfrak{Re}(p) > \alpha$ one can define the Laplace transform of f in p as

$$\widetilde{f}(p) := \int_0^{+\infty} e^{-pt} f(t)\mathrm{d}t \tag{3.2}$$

Indeed it is well-defined since

$$\left| \int_0^{+\infty} e^{-pt} f(t)\mathrm{d}t \right| \leq \int_0^{+\infty} |e^{-pt} f(t)|\mathrm{d}t = \int_0^{c} |e^{-pt} f(t)|\mathrm{d}t + \int_c^{+\infty} |e^{-pt} f(t)|\mathrm{d}t \leq$$

$$\overset{f \in L^1_{loc}(\mathbb{R}^+)}{\leq} \tau(c) + \int_c^{+\infty} e^{-\mathfrak{Re}(p)t} |f(t)|\mathrm{d}t \leq \tau(c) + M \int_c^{+\infty} e^{-(\mathfrak{Re}(p)-\alpha)t}\mathrm{d}t$$

$$= \tau(c) + M \cdot \frac{e^{-(\mathfrak{Re}(p)-\alpha)c}}{\mathfrak{Re}(p)-\alpha} \qquad \in \mathbb{R}$$

For example, the function over $\mathbb{R}^+$ $f(t) := 1$ is in $L^1_{loc}(\mathbb{R}^+)$ and for $\alpha = 0, M = 1, c = 0$ it also satisfies 3.1. Hence for $p \in \mathbb{C}$ such that $\mathfrak{Re}(p) > 0$

$$\widetilde{1}(p) = \int_0^{+\infty} e^{-pt}\mathrm{d}t = \frac{1}{p} \tag{3.3}$$

The function $f(t) := t$ belongs to $L^1_{loc}(\mathbb{R}^+)$ and one can choose any $\alpha > 0$. Indeed $(te^{-\alpha t})' = e^{-\alpha t}(1 - \alpha t)$, hence $te^{-\alpha t}$ has maximum $(\frac{1}{\alpha e})$ for $t = \frac{1}{\alpha}$, and this implies that $\forall t > 0$ $te^{-\alpha t} \leq \frac{1}{\alpha e}$, i.e. $t \leq \frac{e^{\alpha}}{\alpha e}$, and one can choose $c = 0$ and for $\alpha \to 0$ $M = 1$.

$$\widetilde{f}(p) = \int_0^{+\infty} e^{-pt} t\mathrm{d}t = \int_0^{+\infty} \frac{e^{-pt}}{p}\mathrm{d}t = \frac{1}{p^2} \tag{3.4}$$

Finally, for $0 < \beta < 1$, consider $f(t) = \frac{1}{t^\beta}$. $f \in L^1_{loc}(\mathbb{R}^+)$ (notice that it is not true if $\beta = 1$). 3.1 holds for $\alpha = 0$, $M = 1$ and $c = 1$.

Remark 9 *Thanks to the linearity of the integral, the Laplace transform is a linear operator.*

Proposition 3.0.1 *Let $f \in L^1_{loc}(\mathbb{R}^+)$ satisfy 3.1. Then $\widetilde{f}$ is an holomorphic function in $\{z \in \mathbb{C} : \mathfrak{Re}(z) > \alpha\}$.*
Moreover, for $p_0 \in \mathbb{C}$ such that $\mathfrak{Re}(p_0) > \alpha$

$$\frac{d\widetilde{f}}{dp}(p_0) = -\widetilde{(tf(t))}(p_0) \tag{3.5}$$

Proof. Consider $p_0, p \in \mathbb{C}$ such that $\mathfrak{Re}(p_0 + p) > \alpha$. Let Δp denote the difference $p - p_0$. Then

$$\frac{\widetilde{f}(p) - \widetilde{f}(p_0)}{p - p_0} = \frac{\widetilde{f}(p_0 + \Delta p) - \widetilde{f}(p_0)}{\Delta p} = \frac{\int_0^{+\infty} e^{-(p_0+\Delta p)t} f(t)\mathrm{d}t - \int_0^{+\infty} e^{-p_0 t} f(t)\mathrm{d}t}{\Delta p}$$
$$= \int_0^{+\infty} e^{-p_0 t}\, \frac{e^{-\Delta p t} - 1}{\Delta p} f(t)\mathrm{d}t$$

The aim is to prove that $\exists \lim_{p \to p_0} \frac{\widetilde{f}(p) - \widetilde{f}(p_0)}{p - p_0}$ (i.e. $\widetilde{f}$ is holomorphic) and to write it in a practical form (i.e. to prove 3.5). In order to achieve this, one can prove ‡:

$$\lim_{p \to p_0} \frac{\widetilde{f}(p) - \widetilde{f}(p_0)}{p - p_0} = \lim_{p \to p_0} \int_0^{+\infty} e^{-p_0 t}\, \frac{e^{-\Delta p t} - 1}{\Delta p} f(t)\mathrm{d}t \tag{3.6}$$
$$\overset{\ddagger}{=} \int_0^{+\infty} \lim_{p \to p_0} \left(e^{-p_0 t}\, \frac{e^{-\Delta p t} - 1}{\Delta p} f(t) \right) \mathrm{d}t \tag{3.7}$$
$$= -\int_0^{+\infty} e^{-p_0 t} t f(t)\mathrm{d}t = -\widetilde{tf(t)}(p_0) \tag{3.8}$$

Notice that the integral on the right of ‡ makes sense:

$$|e^{-p_0 t} t f(t)| \le e^{-\mathfrak{Re}(p_0)t} t f(t) \le M e^{-\mathfrak{Re}(p_0)t} t e^{\alpha t} = M e^{\overbrace{(-\mathfrak{Re}(p_0) + \alpha)}^{<0} t} t \in L^1(\mathbb{R}^+)$$

‡ will be proved applying Lebesge dominated convergence theorem. To apply that theorem it has just to be proved the the sequence labelled with p is dominated by a function in L^1:

$$\left|e^{-p_0 t}\,\frac{e^{-\Delta p t}-1}{\Delta p}f(t)\right| = e^{-\mathfrak{Re}(p_0)t}\left|\frac{e^{-\Delta p t}-1}{\Delta p}\right||f(t)|$$
$$= e^{-\mathfrak{Re}(p_0)t}\left|\frac{\int_0^1(-\Delta p t e^{-\Delta p t s})\mathrm{d}s}{\Delta p}\right||f(t)| = e^{-\mathfrak{Re}(p_0)t}\left|t\int_0^1 e^{-\Delta p t s}\mathrm{d}s\right||f(t)|$$
$$\leq e^{-\mathfrak{Re}(p_0)t}t\int_0^1|e^{-\Delta p t s}|\mathrm{d}s\,|f(t)| = e^{-\mathfrak{Re}(p_0)t}t\int_0^1 e^{-t\mathfrak{Re}(\Delta p)s}\mathrm{d}s\,|f(t)|$$
$$\stackrel{*}{\leq} e^{-\mathfrak{Re}(p_0)t}t\int_0^1 e^{t\rho s}\mathrm{d}s\,|f(t)| \stackrel{**}{=} tKe^{-\mathfrak{Re}(p_0)t}|f(t)| =: g(t)$$

$*$ follows noticing that one wants to study what happens when $|\Delta p| \to 0$, hence it is possible to choose $\rho \in \mathbb{R}, \rho > 0$ such that $|\Delta p| < \rho$. The integral becomes a constant K and therefore $**$ follows. $g \in L^1(\mathbb{R}^+)$, and the proof is accomplished.

Recall the following property about holomorphic functions:

Proposition 3.0.2 *Let Ω_f, Ω_g be open subsets of $\mathbb{C}$, $f : \Omega_f \to \mathbb{C}$ and $g : \Omega_g \to \mathbb{C}$ be holomorphic functions. If there exists a set $A \subset \Omega_f \cap \Omega_g$ such that*

- *A has an accumulation point in $\Omega_f \cap \Omega_g$*
- *$f\restriction_A = g\restriction_A$*

then $f\restriction_{\Omega_f\cap\Omega_g} = g\restriction_{\Omega_f\cap\Omega_g}$

In particular this proposition implies that if $\widetilde{f}$ is the Laplace transform of a function f, defined for $z \in \mathbb{C}$ such that $\mathfrak{Re}(z) > \alpha$, and g is an holomorphic function defined for $z \in \mathbb{C}$ such that $\mathfrak{Re}(z) > \alpha$, moreover if $\widetilde{f}$ and g coincide when z is real, then they coincide everywhere.

Now take $\gamma \in \mathbb{R}$ and consider the function over $\mathbb{R}^+$ $h(t) = e^{i\gamma t}$. h is bounded, hence it is possible to compute the Laplace transform of h, and it is defined for any $p \in \mathbb{C}$ such that $\mathfrak{Re}(p) > 0$.

$$\widetilde{e^{i\gamma t}}(p) = \int_0^{+\infty} e^{-pt}e^{i\gamma t}\mathrm{d}t = \int_0^{+\infty} e^{-(p-i\gamma)t}\mathrm{d}t = \lim_{L\to+\infty}\left[-\frac{e^{-(p-i\gamma)t}}{p-i\gamma}\right]_0^L = \frac{1}{p-i\gamma}$$

Since sine and cosine are bounded functions, one can compute their Fourier transform for any $p \in \mathbb{C}$ such that $\mathfrak{Re}(p) > 0$. By the linearity of the laplace transform

$$\widetilde{\cos(\gamma t)}(p) + i\widetilde{\sin(\gamma t)}(p) = \widetilde{e^{i\gamma t}}(p) = \frac{1}{p-i\gamma}$$

In particular, for real values of p,

$$\widetilde{\cos(\gamma t)}(\mathfrak{Re}(p)) + i\widetilde{\sin(\gamma t)}(\mathfrak{Re}(p)) = \frac{1}{\mathfrak{Re}(p)-i\gamma} = \frac{1}{\mathfrak{Re}(p)-i\gamma}\cdot\frac{\mathfrak{Re}(p)+i\gamma}{\mathfrak{Re}(p)+i\gamma} = \frac{\mathfrak{Re}(p)+i\gamma}{\mathfrak{Re}(p)^2+\gamma^2}$$

and this, separating the real and the imaginary part, reads

$$\widetilde{\cos(\gamma t)}(\mathfrak{Re}(p)) = \frac{\mathfrak{Re}(p)}{\mathfrak{Re}(p)^2 + \gamma^2}$$
$$\widetilde{\sin(\gamma t)}(\mathfrak{Re}(p)) = \frac{\gamma}{\mathfrak{Re}(p)^2 + \gamma^2}$$

i.e. $\widetilde{\cos(\gamma t)}$ and $\widetilde{\sin(\gamma t)}$ coincide on $\mathbb{R}^+$ respectively with the holomorphic functions $\frac{p}{p^2+\gamma^2}$ and $\frac{\gamma}{p^2+\gamma^2}$ (removing the singularities in $i\gamma, -i\gamma$, they are holomorhic functions on the whole $\mathbb{C}$). Therefore, recalling the remark just after Proposition 3.0.2, one gets

$$\widetilde{\cos(\gamma t)}(p) = \frac{p}{p^2 + \gamma^2} \tag{3.9}$$
$$\widetilde{\sin(\gamma t)}(p) = \frac{\gamma}{p^2 + \gamma^2} \tag{3.10}$$

Now consider the function defined for $t > 0$ as $g(t) := \frac{1}{t^\nu}$, $\nu < 1$. One can compute the Laplace transform of g. For p real

$$\widetilde{g}(\mathfrak{Re}(p)) = \int_0^{+\infty} e^{-\mathfrak{Re}(p)t} \frac{1}{t^\nu} \mathrm{d}t = \frac{1}{\mathfrak{Re}(p)^{1-\nu}} \int_0^{+\infty} e^{-(\mathfrak{Re}(p)t)} \frac{1}{(\mathfrak{Re}(p)t)^\nu} \mathfrak{Re}(p) \mathrm{d}t$$
$$= \frac{1}{\mathfrak{Re}(p)^{1-\nu}} \int_0^{+\infty} e^{-s} \frac{1}{s^\nu} \mathrm{d}s = \frac{1}{\mathfrak{Re}(p)^{1-\nu}} \int_0^{+\infty} e^{-s} s^{(1-\nu)-1} \mathrm{d}s = \frac{\Gamma(1-\nu)}{\mathfrak{Re}(p)^{1-\nu}}$$

where Γ denotes the Gamma function. Notice that $\Gamma(1-\nu)$ is a constant and that the function $\frac{\Gamma(1-\nu)}{p^{1-\nu}}$ can be extended to a holomorphic function over $\mathbb{C} \setminus \{z \in \mathbb{C} : \mathfrak{Re}(z) < 0, \mathfrak{Im}(z) = 0\}$. Since the latter function coincides over $\mathbb{R}^+$ with the Laplace transform that is to be computed, again from the remark just after Proposition 3.0.2, one gets that

$$\widetilde{g}(p) = \frac{\Gamma(1-\nu)}{p^{1-\nu}} \tag{3.11}$$

Remark 10 *Let $f, g \in L^1_{loc}(\mathbb{R}^+)$ satisfy 3.1 with M_f, α_f, c_f and M_g, α_g, c_g respectively. Then for $p \in \mathbb{C}$ such that $\mathfrak{Re}(p) > \max\{\alpha_f, \alpha_g\}$*

$$\widetilde{(f * g)}(p) = \widetilde{f}(p)\widetilde{g}(p) \tag{3.12}$$

Indeed

$$\begin{aligned}\widetilde{(f*g)}(p) &= \int_0^{+\infty} e^{-pt}(f*g)(t)\mathrm{d}t = \int_0^{+\infty}\left(e^{-pt}\int_0^t f(s)g(t-s)\mathrm{d}s\right)\mathrm{d}t \\ &= \int_0^{+\infty}\mathrm{d}s\int_s^{+\infty}\mathrm{d}t\left(e^{-pt}f(s)g(t-s)\right) = \int_0^{+\infty}\mathrm{d}s\left(f(s)e^{-ps}\int_s^{+\infty}e^{-p(t-s)}g(t-s)\mathrm{d}t\right) \\ &= \int_0^{+\infty}\mathrm{d}s\left(f(s)e^{-ps}\int_0^{+\infty}e^{-py}g(y)\mathrm{d}y\right) = \int_0^{+\infty}\mathrm{d}s\left(f(s)e^{-ps}\widetilde{g}(p)\right) \\ &= \widetilde{g}(p)\int_0^{+\infty}f(s)e^{-ps}\mathrm{d}s = \widetilde{f}(p)\widetilde{g}(p)\end{aligned}$$

Thanks to the latter remark it is easy to compute again the Laplace transform of the identity $f(t) = t$ knowing that the Laplace transform of $h(t) = 1$ is $\widetilde{h}(p) = \frac{1}{p}$. Indeed

$$(h*h)(t) = \int_0^t 1\cdot 1\mathrm{d}t = t = f(t)$$

Therefore

$$\widetilde{f}(p) = \widetilde{h*h}(p) = \widetilde{h}(p)\widetilde{h}(p) = \frac{1}{p}\cdot\frac{1}{p} = \frac{1}{p^2}$$

Notice that, for $n \in \mathbb{N}$, define $g_1(t) := (h*h)(t)$ and $g_{n+1} := (g_n * h)(t)$. Then, by induction, $g_n(t) = \frac{t^n}{n!}$. This statement has just been proved for $n = 1$, and if it holds for $n \geq 1$

$$g_{n+1}(t) = \int_0^t g_n(s)\cdot 1\mathrm{d}s = \int_0^t \frac{s^n}{n!}\mathrm{d}s = \frac{t^{n+1}}{(n+1)!}$$

Moreover, for $p \in \mathbb{C}$ such that $\mathfrak{Re}(p) > 0$, $\widetilde{g_n}(p) = \frac{1}{p^{n+1}}$. Indeed, again by induction, the step for $n = 1$ has been proved, and if the claim is true for $n \geq 1$

$$\widetilde{g_{n+1}}(p) = \widetilde{g_n * h}(p) = \widetilde{g_n}(p)\widetilde{h}(p) = \frac{1}{p^n}\cdot\frac{1}{p} = \frac{1}{p^{n+1}}$$

Next theorem will be stated without proof:

Theorem 3.0.1 *Suppose f, g to be functions over $\mathbb{R}^+$ such that it is possible to compute their Laplace transform and $\widetilde{f} = \widetilde{g}$. Then $f = g$ a.e..*

Proposizione 3.0.3 *Let $f \in C^1(\mathbb{R}^+)$ be such that one can compute its Laplace transform. Then, for an admissible p,*

$$\widetilde{f'}(p) = p\widetilde{f}(p) - f(0^-) \tag{3.13}$$

Proof.

$$\widetilde{f'}(p) = \int_0^{+\infty} e^{-pt}f'(t)\mathrm{d}t = \lim_{L\to+\infty}\left[e^{-pt}f(t)\right]_0^L + p\int_0^{+\infty}e^{-pt}f(t)\mathrm{d}t = p\widetilde{f}(p) - f(0^-)$$

3.1 An application to ODE

Consider the problem

$$\begin{cases} f' = \lambda f \\ f(0) = z \end{cases} \tag{3.14}$$

where $\lambda, z \in \mathbb{C}$ are given constants and f is an unknown function defined over $\mathbb{R}^+ \cup \{0\}$ belonging to $C^1(\mathbb{R}^+) \cup C^0(\mathbb{R}^+ \cup \{0\})$ and admitting Laplace transform.
Therefore, applying the Laplace transform and recalling 3.13,

$$p\widetilde{f}(p) - z = \lambda \widetilde{f}(p) \tag{3.15}$$

Hence $\widetilde{f}(p) = \frac{z}{p-\lambda}$, and thanks to Theorem 3.0.1 and the result about the Laplace transform of $t \mapsto e^{\lambda t}$, one gets

$$f(t) = e^{\lambda t} z \tag{3.16}$$

The above problem can be modified in

$$\begin{cases} f' = \lambda f + g \\ f(0) = z \end{cases} \tag{3.17}$$

where f, λ, z are as before and g is a function over $\mathbb{R}^+$ admitting Laplace transform. Applying again the Laplace transform one gets

$$p\widetilde{f}(p) - z = \lambda \widetilde{f}(p) + \widetilde{g} \tag{3.18}$$

$$\widetilde{f}(p) = \frac{z}{p-\lambda} + \widetilde{g}\frac{1}{p-\lambda} \tag{3.19}$$

Hence, thanks to the same observations as before and to the linearity, one gets a formula which is usually referred to as the variation of constants formula:

$$f(t) = e^{\lambda t} z + (g * e^{\lambda t}) = e^{\lambda t} z + \int_0^t e^{\lambda(t-s)} g(s) \mathrm{d}s \tag{3.20}$$

3.2 Inverse Laplace transform

A heuristic argument. Given a function $\widetilde{f}$, one might want to find a function f over $\mathbb{R}^+$ such that for some $p \in \mathbb{C}$

$$\widetilde{f}(p) = \int_0^{+\infty} e^{-pt} f(t) \mathrm{d}t \tag{3.21}$$

Define for $t \in \mathbb{R}$

$$\varphi(t) := \begin{cases} f(t) & \text{if} \quad t \in \mathbb{R}^+ \\ 0 & \text{if} \quad t \in \mathbb{R} \setminus \mathbb{R}^+ \end{cases} \tag{3.22}$$

Let $x := \mathfrak{Re}(p)$ and $y := \mathfrak{Im}(p)$. Now

$$\widetilde{f}(p) = \int_0^{+\infty} e^{-xt} e^{-iyt} f(t)\mathrm{d}t = \int_{-\infty}^{+\infty} e^{-iyt} \left(e^{-xt}\varphi(t)\right)\mathrm{d}t$$

i.e. $\widetilde{f}$ might be the Fourier transform of $t \mapsto e^{-xt}\varphi(t)$. If it were so, one might be able to apply the inverse Fourier transform, getting

$$\varphi(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{(x+iw)t}\widetilde{f}(x+iw)\mathrm{d}w = \frac{1}{2\pi i}\int_{\Gamma_x} e^{pt}\widetilde{f}(p)\mathrm{d}p \tag{3.23}$$

where $\Gamma_x = \{x + iw \in \mathbb{C} : w \in \mathbb{R}\}$. Then one would expect $\frac{1}{2\pi i}\int_{\Gamma_x} e^{pt}\widetilde{f}(p)\mathrm{d}p$ to be zero when $t \in \mathbb{R} \setminus \mathbb{R}^+$ and for $t \in \mathbb{R}^+$

$$f(t) = \frac{1}{2\pi i}\int_{\Gamma_x} e^{pt}\widetilde{f}(p)\mathrm{d}p \tag{3.24}$$

Actually the inverse Laplace transform can be computed as an improper Riemann integral: considering x such that $\widetilde{f}$ is defined and holomorphic for all the $p \in \mathbb{C}$ such that $\mathfrak{Re}(p) > x$,

$$f(t) = \lim_{L\to+\infty} \frac{1}{2\pi i}\int_{x-iL}^{x+iL} e^{pt}\widetilde{f}(p)\mathrm{d}p \tag{3.25}$$