1. Serie di FOURIER

2. Integrale di Fourier   Trasformata di Fourier

3. Trasformata di Laplace

4. Trasformata di Mellin

Applicazioni : Equazioni differenziali ordinarie (ODE)
Equazioni alle derivate parziali (PDE)
Equazioni integrali ed integro differenziali

Funzioni speciali, Spazi di Hilbert, basi ortonormali

Teoria dei segnali $\boxed{\text{ANALISI FUNZIONALE}}$

È uno strumento che si applica solo ai problemi LINEARI (o non lineari quadratici)

Problemi ellittici

Problemi di evoluzione
- problemi parabolici
- problemi iperbolici

Equazione di Schrödinger

Metodi asintotici

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

$$\left(a_0, a_1, a_2, a_3, \ldots ; b_1, b_2, b_3, \ldots \right)$$ successione di numeri

$$\left(a_0, a_1, b_1, a_2, b_2, a_3, b_3, \ldots \right) \in \mathbb{R}^{\infty}$$

$$\left(a_0, a_1, b_1, a_2, b_2, \ldots, a_N, b_N, 0, 0, 0, \ldots \right) \in \mathbb{R}^{\infty}$$

$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

$$\left(S_0(x), S_1(x), S_2(x), S_3(x), \ldots \right)$$

successione di funzioni periodiche

$\{ S_N(x) \}$ quando converge e che tipo di convergenza?

questo è il problema centrale delle serie di Fourier

Trasformate Integrali

Serie numeriche , serie di funzioni

$a_1 + a_2 + a_3 + a_4 + \dots \overset{N}{=} \sum_{i=1}^{\infty} a_i$ 　　　 i indice

$a_i \in \mathbb{R}$

la successione delle "ridotte della serie"

$s_1, s_2, s_3, s_4, \dots$

$s_n = a_1 + a_2 + \dots + a_n \overset{N}{=} \sum_{i=1}^{n} a_i$

Utilizzare la nozione di limite ~~di una~~ classico di una successione (se esiste)

$$\sum_{i=1}^{\infty} a_i \overset{\text{def}}{=} \lim_{n \to \infty} s_n \quad (\text{se esiste})$$

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limite di una successione secondo Cesáro (Ernesto)

$s_1, s_2, \dots \xrightarrow{\text{cesàro}} s$

$$\frac{s_1 + s_2 + \dots + s_n}{n} \longrightarrow s$$

Teorema: Se una successione converge (classicamente) ad un limite L, allora converge anche secondo Cesáro allo stesso limite (ma non viceversa)

Dimostr.: sia $\{a_n\}$ la successione convergente ad L: per ogni $\varepsilon > 0$ si ha $|a_n - L| < \varepsilon$ $\forall n \geq n_\varepsilon$. Allora per $n \geq n_\varepsilon$

$$\left| \frac{a_1 + a_2 + \dots + a_n}{n} - L \right| =$$

$$= \left| \frac{a_1 + a_2 + \dots + a_{n_\varepsilon}}{n} + \frac{a_{n_\varepsilon+1} - L + a_{n_\varepsilon+2} - L + \dots + a_n - L}{n} + \frac{n_\varepsilon}{n} L \right|$$

$$\leq \left| \frac{a_1 + a_2 + \dots + a_{n_\varepsilon}}{n} \right| + \frac{(n - n_\varepsilon)\varepsilon}{n} + \frac{n_\varepsilon}{n} |L|$$

$n \geq \bar{n}_\varepsilon > n_\varepsilon$

$$< \frac{\varepsilon}{2} + \varepsilon + < \frac{\varepsilon}{2}$$

Viceversa consideriamo l'esempio della successione

$$1, -1, 1, -1, 1, -1, \dots$$

che non converge. Le somme di Cesáro sono

$$1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7} \longrightarrow 0$$

Punto di vista funzionale

Denotiamo con $\mathbb{R}^\infty$ l'insieme di tutte le successioni, che è anche lo spazio prodotto di $\mathbb{N}$ copie di $\mathbb{R}$

$\mathbb{R}^\infty = \prod_{i=1}^{\infty} \mathbb{R}$. È uno spazio vettoriale (sui reali)

Fissato un numero $p \geq 1$ consideriamo il sottospazio, che denoteremo con ℓ^p, formato da tutte le successioni di $\mathbb{R}^\infty$ tali che

$$\sum_{i=1}^{\infty} |a_i|^p < +\infty$$

$\ell^p \subset \mathbb{R}^\infty$: per ogni $p \geq 1$ ℓ^p è uno spazio vettoriale in cui introduciamo una <u>norma</u>

$$a \in \ell^p \qquad |a|_p = \left(\sum_{i=1}^{\infty} |a_i|^p \right)^{1/p}$$

Esercizio - Verificare che $|a|_p$ è una norma -

Esercizio - ℓ_p è uno spazio di Banach per ogni $p \geq 1$

Sia ℓ^∞ l'insieme di tutte le successioni $a \in \mathbb{R}^\infty$ limitate ($|a_i| \leq \gamma < +\infty$ per ogni i) - In ℓ^∞ introduciamo la norma

$$|a|_\infty = \sup_i |a_i|$$

Esercizio: $|a|_\infty$ è una norma.

Esercizio: ℓ^∞ è uno spazio di Banach -

Sia c_0 l'insieme delle successioni di $\mathbb{R}^\infty$ tali che $\lim_{n \to \infty} a_n = 0$

Teorema: $\forall p \geq 1 \quad \ell^p \subset c_0$

Nota bene: $c_0 \subset \ell^\infty$

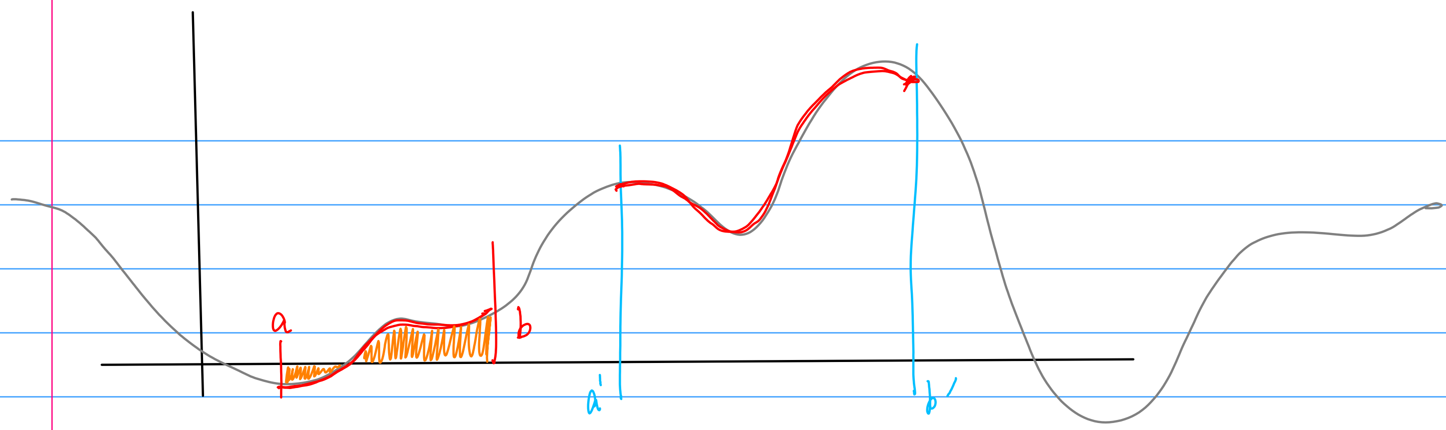
Osservazione: Sia $\gamma = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$

$\gamma \in \ell^p$ per $p > 1$ ma $\gamma \notin \ell^1$

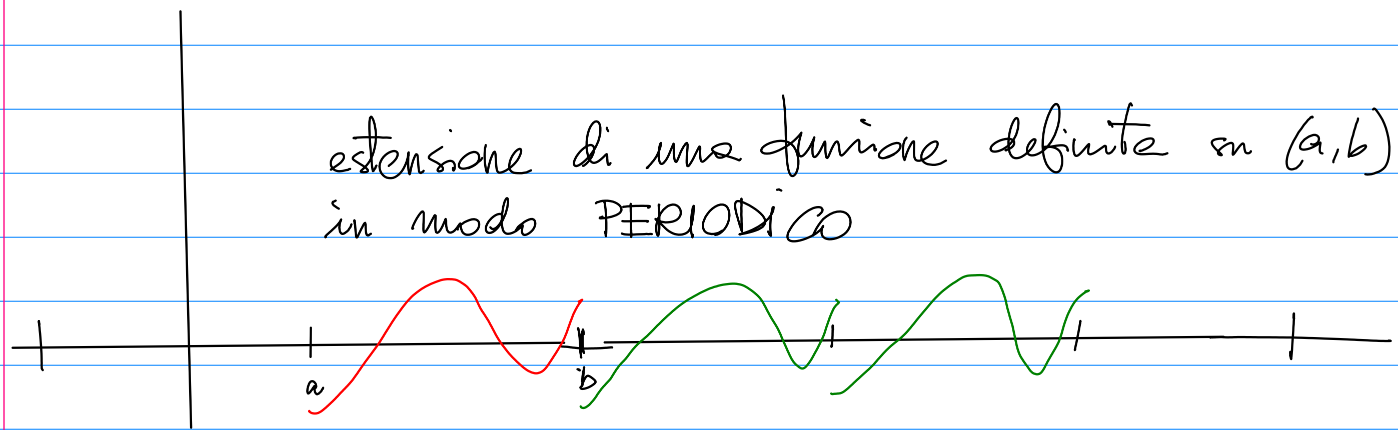
ℓ^2 ha una struttura di spazio di Hilbert con prodotto scalare $(a, b) = \sum_{i=1}^{\infty} a_i b_i$

Esercizio: estendere a $\mathbb{C}^\infty$: $\ell^p(\mathbb{C})$

Per le serie di funzioni il discorso è più complesso in quanto ci sono molti modi per definire la convergenza di successioni di funzioni (e quindi di serie di funzioni).



estensione di una funzione definita su (a,b)
in modo PERIODICO



[funzioni periodiche, periodo di una funzione periodica.

Discontinuità di prima e seconda specie.

$\sin t$, $\cos t$, $\sin(2t)$, $\cos(2t)$
$\sin(nt)$, $\cos(nt)$

$$\frac{a_0}{2} + a_1 \cos t + b_1 \sin t + a_2 \cos(2t) + b_2 \sin(2t) + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

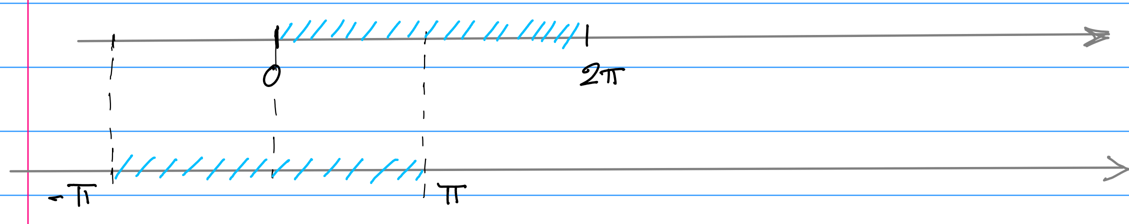
$a_0 \in \mathbb{R} \quad a \in \mathbb{R}^\infty, \; b \in \mathbb{R}^\infty \qquad (a_0, a_1, b_1, a_2, b_2, \ldots) \in \mathbb{R}^\infty$

Questione: studiare la CONVERGENZA di questa serie di funzioni (definite per $t \in \mathbb{R}$)

Quali tipi di convergenze è utile considerare?

1. convergenza puntuale (è la più immediata - prima)
2. convergenza uniforme (è una convergenza più forte)
3. convergenza in L^p (L^2) (dopo Lebesgue)
4. ....
5. ....

In particolare la convergenza in L^2 (convergenza in media quadratica)



$$\int_a^{a+2\pi} \cos(nt)\cos(mt)\,dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \geq 1 \\ 2\pi & n = m = 0 \end{cases}$$

$$\int_{a}^{a+2\pi} \sin(nt)\cos(mt)\,dt = 0$$

$$\int_{a}^{a+2\pi} \sin(nt)\sin(mt)\,dt = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases} \qquad n, m \geq 1$$

$T \qquad \cos\left(n\frac{2\pi}{T}t\right) \qquad \sin\left(n\frac{2\pi}{T}t\right)$

$\frac{2\pi}{T} = \omega$

$\frac{1}{T} = \nu$

$\boxed{\omega = 2\pi\nu}$

$$\int_{a}^{a+T} \cos(n\omega t)\cos(m\omega t)\,dt = \begin{cases} 0 & n \neq m \\ \frac{T}{2} & n = m \geq 1 \\ T & n = m = 0 \end{cases}$$

$$(a_0, a_1, b_1, a_2, b_2, \dots) \in \mathbb{R}^{\infty}$$

$$S_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k\cos(kt) + b_k\sin(kt)$$

$$S_n \xrightarrow{?} f$$

$$\boxed{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos(nx)\,dx}$$

$$\boxed{b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin(nx)\, dx}$$

Quando partindo da f

$$(a_0, a_1, b_1, a_2, b_2, \ldots\ldots)$$

$$s_n \xrightarrow{?} g \;(= f)$$

$H \quad \{e_1, e_2, e_3, e_4, \dots \}$ <u>ortonormale</u>

$x \in H \qquad \langle x, e_k \rangle$

$$\left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\|^2 =$$

$$= \left\langle x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k , \ x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\rangle =$$

$$= \|x\|^2 - \sum_{k=1}^{n} \langle x, e_k \rangle^2 \geq 0$$

$$\sum_{k=1}^{n} \langle x, e_k \rangle^2 \leq \|x\|^2 \qquad \underline{\text{Bessel}}$$

$$\left\| x - \sum_{k=1}^{n_\varepsilon} \xi_k^{(\varepsilon)} e_k \right\|^2 < \varepsilon^2$$

$$\|x\|^2 - 2 \sum_{k=1}^{n_\varepsilon} \xi_k^{(\varepsilon)} \langle x, e_k \rangle + \sum_{k=1}^{n_\varepsilon} \xi_k^{(\varepsilon)2} < \varepsilon^2$$

$$\|x\|^2 - \sum_{k=1}^{n_\varepsilon} \langle x, e_k \rangle^2 + \sum_{k=1}^{n_\varepsilon} \left(\xi_k^{(\varepsilon)} - \langle x, e_k \rangle \right)^2 < \varepsilon^2$$

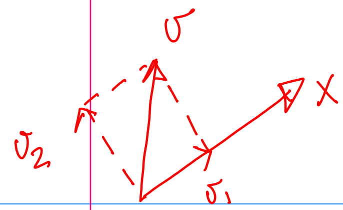
$$\Longrightarrow \boxed{\|x\|^2 = \sum_{k=1}^{\infty} \langle x, e_k \rangle^2}$$

$$v_n = \sum_{k=1}^{n} \langle x, e_k \rangle e_k \qquad \|v_{n+p} - v_n\|^2 \qquad \|v_n - v\|^2 \to 0$$

$$= \sum_{k=n+1}^{n+p} \langle x, e_k \rangle^2 \longrightarrow 0$$

$$v = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$$

$$\underbrace{\langle v, x \rangle = \|x\|^2}_{A} \qquad \|v\| = \|x\|$$



$v = \lambda x \Rightarrow \underline{\lambda = 1} \quad v = x$

$v_1 = \frac{\langle v, x \rangle}{\|x\|^2} x \qquad \langle v - v_1, x \rangle =$

$= \langle v, x \rangle - \langle v, x \rangle = 0$

$v_2 = v - v_1$

$\langle v_1, v_2 \rangle = \langle v, v_1 \rangle - |v_1|^2 \qquad v = v_1 + v_2$

$= \frac{\langle v, x \rangle^2}{\|x\|^2} - \frac{\langle v, x \rangle^2}{\|x\|^2} = 0$

$\Leftarrow \quad v_1 \equiv x \qquad \|v\|^2 = \|v_1\|^2 + \|v_2\|^2 \qquad \|v_2\| = 0$

$v \equiv x$

$(\cos x)^n$ $(\sin x)^n$

$\cos x$ $\sin x$

$\cos^2 x = \frac{1+\cos 2x}{2}$ $\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos^3 x = \cos x \, \frac{1+\cos 2x}{2} =$

$= \frac{1}{2}\cos x + \frac{1}{2}\cos x \cos 2x =$

$= \frac{1}{2}\cos x + \frac{\cos(3x)+\cos x}{4}$

$= \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ 1 3

$\cos^n x = \sum a_k \cos(kx) + b_k \sin(kx)$
$\sin^n x$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

$\cos x = \frac{e^{ix}+e^{-ix}}{2}$ $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$

$$\cos^n x \sin^m x = \sum_{k=0}^{N} a_k \cos(kx) + b_k \sin(kx)$$

$P(x, y)$ $\qquad$ $P(\cos x, \sin x)$

polinomio trigonometrico

$$P(\cos x, \sin x) = \sum_{k=0}^{N} a_k \cos(kx) + b_k \sin(kx)$$

$$\varphi(x) = 1 + \cos\left(x - \frac{a+b}{2}\right) - \cos\frac{a-b}{2}$$

$0 < a < b < 2\pi$

$\psi_n(x) = \varphi(x)^n$ polinomio trigonometrico

$f \in L^1(0, 2\pi)$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)\,dx$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)\,dx$$

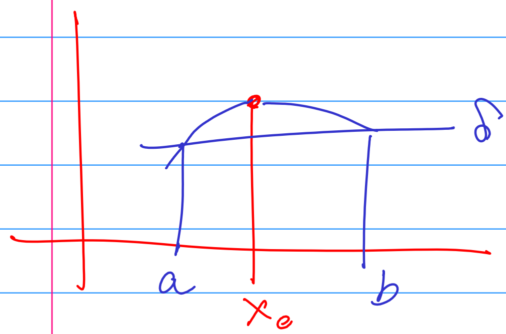
Ipotesi: $a_k = 0 \quad k = 0, 1, 2 \dots \qquad b_k = 0 \quad k = 1, 2, \dots$

Tesi $f = 0$ quasi ovunque (rispetto Lebesgue)

1° passo: supponiamo $f \in C([0, 2\pi])$

supp. che $f \neq 0$

cons. $\exists\ [a,b]$ dove $f \geq \delta > 0$



$[a,b] \subset [0, 2\pi]$

$f \geq \delta$

$$\underbrace{0 = \int_0^{2\pi} f(x)\,\psi_n(x)\,dx} = \int_a^b f(x)\,\psi_n(x)\,dx + \int_{[0,2\pi]\setminus[a,b]} f(x)\,\psi_n(x)\,dx$$

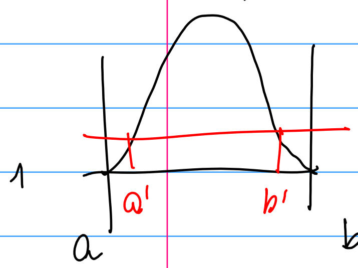
$$\int_a^b f(x)\,\psi_n(x)\,dx + \int_{[0,2\pi]\setminus[a,b]} f(x)\,\psi_n(x)\,dx = 0$$

$$\left| \int_{[0,2\pi]\setminus[a,b]} f(x)\,\psi_n(x)\,dx \right| \leq \int_{[0,2\pi]\setminus[a,b]} |f(x)|\,|\psi_n(x)|\,dx$$

$|\psi_n(x)| \leq 1$

$$\leq \int_0^{2\pi} |f(x)|\,dx$$

$$\int_a^b f(x)\,\varphi_n(x)\,dx \geq \delta \int_a^b \varphi_n(x)\,dx$$



$$\geq \delta \int_{a'}^{b'} \varphi_n(x)\,dx \geq \delta\,(b'-a')\,(1+\mu)^n$$

2º passo $\quad f \in L^1(0,2\pi)$

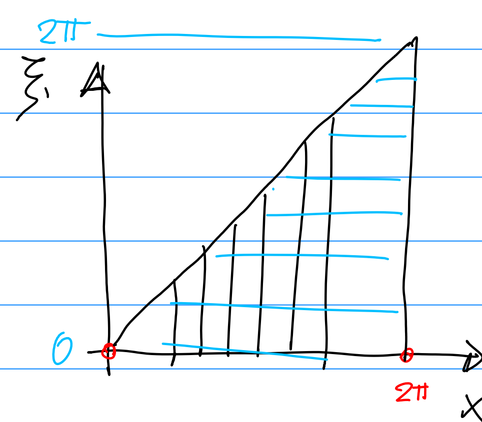
$$x \longrightarrow \int_0^x f(\xi)\,d\xi =: F(x) \qquad |F(x)| \leq \|f\|_1$$

$$F(x) \longrightarrow \alpha_k \quad \beta_k$$

$$\alpha_k = \frac{1}{\pi} \int_0^{2\pi} F(x) \cos(kx)\,dx =$$

$$= \frac{1}{\pi} \int_0^{2\pi} \left\{ \int_0^x f(\xi)\,d\xi \right\} \cos(kx)\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} d\xi\, f(\xi) \int_\xi^{2\pi} \cos(kx)\,dx$$



$$g(x,\xi) = f(\xi)\cos(kx)$$

$$\frac{\sin(kx)}{k}\Bigg]_{\xi}^{2\pi} = -\frac{\sin(k\xi)}{k} \qquad k \geq 1$$

$$\boxed{\alpha_0 = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\,(2\pi - \xi)\,d\xi = 2\pi a_0 - \frac{1}{\pi}\int_0^{2\pi} f(\xi)\,\xi\,d\xi}$$

$$\boxed{\alpha_k = -\frac{1}{\pi}\int_0^{2\pi} d\xi\, f(\xi)\,\frac{\sin(k\xi)}{k} = -\frac{b_k}{k}} \qquad k \geq 1$$

$$\beta_k = \frac{1}{\pi}\int_0^{2\pi} F(x)\,\sin(kx)\,dx = \qquad k \geq 1$$

$$= \frac{1}{\pi}\int_0^{2\pi} F(x)\left[-\frac{\cos(kx)}{k}\right]' dx =$$

$$= \frac{1}{\pi}\left\{\left[-F(x)\,\frac{\cos(kx)}{k}\right]_0^{2\pi} + \int_0^{2\pi} F'(x)\,\frac{\cos(kx)}{k}\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-F(2\pi)\,\frac{1}{k} + \frac{1}{k}\int_0^{2\pi} f(x)\cos(kx)\,dx\right\}$$

$$\cos(2k\pi) = 1$$

$$\boxed{\beta_k = -\frac{1}{\pi}F(2\pi)\,\frac{1}{k} + \frac{1}{k}\,a_k}$$

$a_k = 0 \quad k \geq 0 \qquad b_k = 0 \quad k \geq 1$

$\alpha_k = 0 \quad k \geq 1$

$\Rightarrow$

$$\alpha_0 = -\frac{1}{\pi} \int_0^{2\pi} f(\xi) \cdot \xi \, d\xi$$

$$\beta_k = -\frac{1}{\pi} F(2\pi) \frac{1}{k} = 0$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx = \frac{F(2\pi)}{\pi} = 0$$

$F(x) - \alpha_0 = 0$ ovunque

$F(x) =$ costante

$\Rightarrow f(x) = 0$ quasi ovunque

Conseguenze -

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(kx)}{\sqrt{\pi}}, \dots, \frac{\sin(kx)}{\sqrt{\pi}}, \dots \right\}$$

sistema ortonormale in $L^2(0, 2\pi)$

base ortonormale

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

se $f \in L^2(0,2\pi)$ $\quad L^2$ $\quad L^2(0,2\pi) \subset L^1(0,2\pi)$

H $\quad \{e_1, e_2, \dots\}$ base ortonormale

Hilbert

$$x \in H \qquad \boxed{x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n} \; *$$

$$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \qquad \text{Parseval}$$

$$\sum_{n=1}^{N} |\langle x, e_n \rangle|^2 \leq \|x\|^2 \quad \text{Disuguaglianza di Bessel}$$

$$f(x) \doteq \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

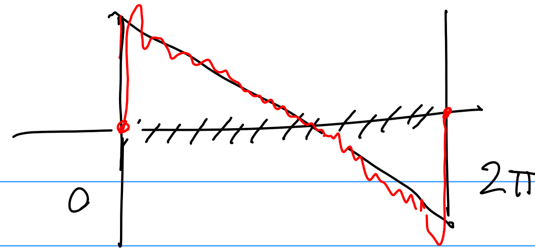
$$S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} a_k \cos(kx) + b_k \sin(kx)$$

$$\int_0^{2\pi} |f(x) - S_N(x)|^2 dx \longrightarrow 0$$

$$\|f\|^2_{L^2} = \int_0^{2\pi} |f(x)|^2 dx = \left(\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \right) \pi$$

$$\frac{a_0^2}{2} + \sum_{k=1}^{N} (a_k^2 + b_k^2) \leq \frac{1}{\pi} \int_0^{2\pi} |f(x)|^2 dx$$

$$f(x) = \frac{\pi - x}{2}$$



$$f \longrightarrow \begin{array}{ll} a_k = 0 & k \geq 0 \\ b_k = \frac{1}{k} & k \geq 1 \end{array}$$

Esercizio

$$\int_0^{2\pi} \frac{(\pi - x)^2}{4} dx = \pi \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\int_0^{2\pi} (\pi - x)^2 dx = 4\pi \sum_{k=1}^{\infty} \frac{1}{k^2}$$

$$\|$$

$$-\frac{(\pi - x)^3}{3} \Bigg|_0^{2\pi} = \frac{\pi^3}{3} + \frac{\pi^3}{3} = \frac{2}{3}\pi^3$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2}{3} \frac{\pi^3}{4\pi} = \frac{\pi^2}{6}$$

$$\frac{\pi - x}{2} \underset{L^2}{\dot{=}} \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \quad *$$

$$g(x) = (1 - \cos x)\frac{\pi - x}{2}$$

$$a_k = -\frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\cos x \cos(kx)\,dx =$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\pi - x}{2}\left[\cos((k+1)x) + \cos((k-1)x)\right]dx$$

$$= 0$$

$$b_k = \frac{1}{k} - \frac{1}{\pi}\int_0^{2\pi}\frac{\pi - x}{2}\sin(kx)\cos x\,dx =$$

$$= \frac{1}{k} - \frac{1}{2\pi}\int_0^{2\pi}\frac{\pi - x}{2}\left(\sin((k+1)x) + \sin((k-1)x)\right)$$

$$= \frac{1}{k} - \frac{1}{2}\left(\frac{1}{k+1} + \frac{1}{k-1}\right) \qquad k \geq 2$$

$$b_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\boxed{\begin{aligned} b_1 &= \frac{3}{4} \\ b_k &= -\frac{1}{k(k^2-1)} \qquad k \geq 2 \end{aligned}}$$

$$g'(x) = \sin x\frac{\pi - x}{2} - \frac{1}{2}(1 - \cos x)$$

$$g'(x) = \frac{(\pi - x)\sin x - 1 + \cos x}{2}$$

$$g'(0) = 0 = g'(2\pi)$$

$$\frac{3}{4}\sin x - \sum_{k=2}^{\infty} \frac{\sin x}{k(k^2-1)} = g(x)$$

converge uniformemente.

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k}$$

$$(1-\cos x)\, S_n(x) = \sum_{k=1}^{n} \frac{\sin(kx)}{k} - \sum_{k=1}^{n} \frac{\sin(kx)\cos x}{k} =$$

$$= \sum_{k=1}^{n} \frac{\sin(kx)}{k} - \frac{1}{2}\sum_{k=1}^{n} \frac{\sin(k+1)x}{k} - \frac{1}{2}\sum_{k=2}^{n} \frac{\sin(k-1)x}{k} =$$

$$= \sum_{k=1}^{n} \frac{\sin(kx)}{k} - \frac{1}{2}\sum_{k=2}^{n+1} \frac{\sin(kx)}{k-1} - \frac{1}{2}\sum_{k=1}^{n-1} \frac{\sin(kx)}{k+1} =$$

$$= \underbrace{\frac{3}{4}\sin x - \sum_{k=2}^{n} \frac{\sin(kx)}{k(k^2-1)}}_{\downarrow \text{ uniform. } g(x)} \underbrace{- \frac{1}{2}\frac{\sin((n+1)x)}{n} + \frac{1}{2}\frac{\sin(nx)}{n+1}}_{\downarrow\, 0 \quad n\to\infty \text{ uniformemente}}$$

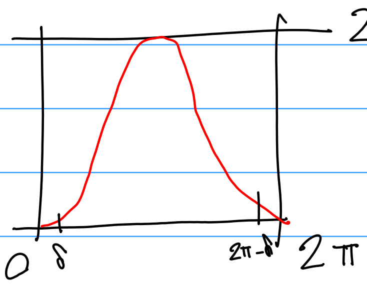
quindi

$$(1-\cos x)\, S_n(x) \xrightarrow[n\to\infty]{} (1-\cos x)\frac{\pi - x}{2}$$

uniformemente

$n \geq n_\varepsilon$ $\quad (1-\cos x)\left|S_n(x) - \frac{\pi-x}{2}\right| < \varepsilon$

$1-\cos x$



$\forall x \in [\delta, 2\pi-\delta] \quad 1-\cos x \geq 1-\cos\delta > 0$

$\forall x \in [\delta, 2\pi-\delta] \quad (1-\cos\delta)\left|S_n(x) - \frac{\pi-x}{2}\right| < \varepsilon$

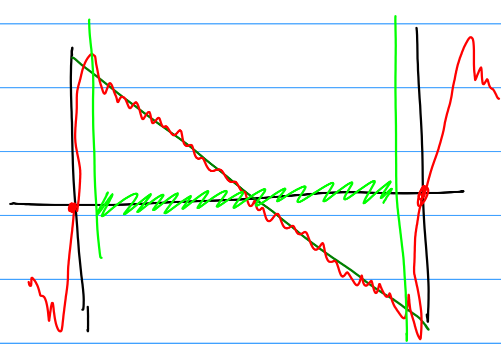
$n \geq n_\varepsilon$ cioè

$$\left|S_n(x) - \frac{\pi-x}{2}\right| < \varepsilon \qquad \forall n \geq n_\varepsilon, \ \forall x \in [\delta, 2\pi-\delta]$$

Abbiamo la convergenza uniforme per

$$\frac{\pi-x}{2} = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

in ogni intervallo $[\delta, 2\pi-\delta] \subset]0, 2\pi[$ $\qquad \delta > 0$



$$h(x) = \begin{cases} 0 & x = 0 \\ \frac{\pi-x}{2} & x \in]0,2\pi[\\ 0 & x = 2\pi \end{cases}$$

Se $f(x)$ è continua a tratti, con derivata continua a tratti (in un numero finito)

Siano $\xi_1, \xi_2, \dots, \xi_m$ i punti di discontinuità di f con $f(\xi_i) = \dfrac{f(\xi_i^+)+f(\xi_i^-)}{2}$ (i limiti sinistro e destro):

Consideriamo la funzione

$$\tilde{f}(x) = f(x) - \sum_{i=1}^{m} h(x-\xi_i)\,\frac{f(\xi_i^+)-f(\xi_i^-)}{2}$$

$$\tilde{f}(\xi_i^-) = f(\xi_i^-) - \sum_{j\neq i} h(\xi_i-\xi_j)\frac{f(\xi_j^+)-f(\xi_j^-)}{2} + \frac{f(\xi_i^+)-f(\xi_i^-)}{2}$$

$$= f(\xi_i) - \sum_{j\neq i} h(\xi_i-\xi_j)\frac{f(\xi_j^+)-f(\xi_j^-)}{2}$$

$$\tilde{f}(\xi_i^+) = f(\xi_i^+) - \sum_{j\neq i} h(\xi_i-\xi_j)\frac{f(\xi_j^+)-f(\xi_j^-)}{2} - \frac{f(\xi_i^+)-f(\xi_i^-)}{2}$$

$$= f(\xi_i) - \sum_{j\neq i} h(\xi_i-\xi_j)\frac{f(\xi_j^+)-f(\xi_j^-)}{2}$$

$\tilde{f}(x)$ è continua nei punti ξ_i, quindi è continua dappertutto.

$$S_n(\tilde{f})(x) \longrightarrow \tilde{f}(x) \quad \underline{\text{uniformemente}}$$

$$S_n(f)(x) = S_n(\tilde{f})(x) + \sum_{i=1}^{m} \frac{f(\xi_i^+)-f(\xi_i^-)}{2} S_n(h(x-\xi_i))$$

$$\Big\downarrow \text{unif.} \qquad\qquad \tilde{f}(x) \qquad\qquad \sum_{i=1}^{m} \frac{f(\xi_i^+) - f(\xi_i^-)}{2}\, h(x-\xi_i)$$

uniform. su insiemi distanti strettamente da ξ_i

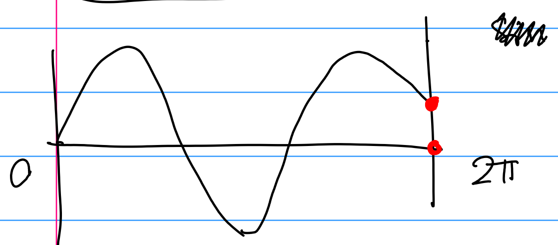
Lo stesso tipo di convergenza abbiamo per

$$S_n(f)(x) \longrightarrow f(x) \qquad \forall x$$

uniform. lontano da ξ_i

$\alpha \notin \mathbb{N}$

$$\begin{matrix} \sin(\alpha x) & \sinh(\alpha x) & e^{\alpha x} \\ \cos(\alpha x) & \cosh(\alpha x) & e^{-\alpha x} \end{matrix}$$



$\sin(\alpha x)$

$$2\int_0^{2\pi} \sin(\alpha x)\cos(kx)\,dx = \int_0^{2\pi} \Big[\sin\big((\alpha+k)x\big) + \sin(\alpha-k)x\Big]\,dx$$

$$= -\frac{\cos(\alpha+k)x}{\alpha+k}\Bigg|_0^{2\pi} - \frac{\cos(\alpha-k)x}{\alpha-k}\Bigg|_0^{2\pi}$$

$$= -\left[\frac{\cos((\alpha+k)2\pi)}{\alpha+k} - \frac{1}{\alpha+k} + \frac{\cos((\alpha-k)2\pi)}{\alpha-k} - \frac{1}{\alpha-k}\right]$$

$$= -\frac{\cos(2\pi\alpha)}{\alpha+k} + \frac{1}{\alpha+k} - \frac{\cos(2\pi\alpha)}{\alpha-k} + \frac{1}{\alpha-k}$$

$$= -\frac{2\alpha\cos(2\pi\alpha)}{\alpha^2-k^2} + \frac{2\alpha}{\alpha^2-k^2}$$

$$a_k = \frac{\alpha}{\pi}\left[\frac{1-\cos(2\pi\alpha)}{\alpha^2-k^2}\right] = \frac{2\alpha\sin^2\pi\alpha}{\pi(\alpha^2-k^2)}$$

$$\pi 2 b_k = 2\int_0^{2\pi}\sin(\alpha x)\sin(kx)\,dx = \int_0^{2\pi}\left\{\cos((\alpha-k)x) - \cos((\alpha+k)x)\right\}$$

$$= \left.\frac{\sin((\alpha-k)x)}{\alpha-k}\right|_0^{2\pi} - \left.\frac{\sin((\alpha+k)x)}{\alpha+k}\right|_0^{2\pi} =$$

$$= \frac{\sin(2\pi\alpha)}{\alpha-k} - \frac{\sin(2\pi\alpha)}{\alpha+k} = \frac{2k\sin(2\pi\alpha)}{\alpha^2-k^2}$$

$$\sin(\alpha x) = \sum_{k=0}^{\infty}\frac{2\alpha\sin^2\pi\alpha}{\pi(\alpha^2-k^2)}\cos kx + \sum_{k=1}^{\infty}\frac{2k\sin\pi\alpha\cos\pi\alpha}{\pi(\alpha^2-k^2)}\sin(kx)$$

$$= \frac{2\sin\pi\alpha}{\pi}\left[\frac{\sin(\pi\alpha)}{\alpha} + \sum_{k=1}^{\infty}\frac{\alpha\cos(kx)\sin(\pi\alpha) + k\sin(kx)\cos(\pi\alpha)}{\alpha^2-k^2}\right]$$

$$f \in L^2(0,2\pi) \longrightarrow a_k, b_k \in \mathbb{R}$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos kx + b_k \sin kx \right)$$

convergenza in L^2

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} \left(a_k^2 + b_k^2 \right) = \frac{1}{\pi} \int_0^{2\pi} f^2(x)\, dx$$

$$L^2 \longleftrightarrow \ell^2$$

$$\longrightarrow \qquad a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx \quad \ldots$$

$$\longleftarrow \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

H Hilbert separabile $\{e_1, e_2, \ldots \quad \ldots\}$

$x \in H$ $\qquad \xi_k = \langle x, e_k \rangle \qquad \sum \xi_k^2 < +\infty$

$$x \in H \longrightarrow \{\xi_k\} \in \ell^2$$

$$\{\xi_k\} \in \ell^2$$

$$x_N = \sum_{k=1}^{N} \xi_k e_k \xrightarrow[H]{} x$$

$$f \in L^2 \implies \underbrace{a_k \to 0 \quad b_k \to 0}_{\text{Riemann - Lebesgue}} \quad k \to \infty$$

$$\Downarrow$$

$$f \in L^1 \implies a_k \to 0 \quad b_k \to 0 \quad k \to \infty$$

$f \in L^1$, $\varepsilon > 0$ $\exists\, g \in L^2$ tale che

$$\| g - f \|_{L^1} < \varepsilon$$

$$f \longrightarrow a_k(f),\ b_k(f)$$

$$g \longrightarrow a_k(g),\ b_k(g)$$

$$\begin{matrix} a_k(g) \to 0 \\ b_k(g) \to 0 \end{matrix} \quad k \to \infty$$

$$|a_k(f)| \leq |a_k(g)| + |a_k(f) - a_k(g)|$$

$$|a_k(f) - a_k(g)| = \left| \frac{1}{\pi} \int_0^{2\pi} \big(f(x) - g(x)\big) \cos kx \, dx \right|$$

$$\leq \frac{1}{\pi} \| f - g \|_{L^1}$$

$$|a_k(f)| \leq |a_k(g)| + \frac{\varepsilon}{\pi}$$

$$\limsup_k |a_k(f)| \leq \frac{\varepsilon}{\pi}$$

CVD.

$$L^1 \longrightarrow C_0$$

Spazi di Banach

1. Banach-Steinhaus
2. Mappa aperta
3. Hahn-Banach

$$f \in L^2(0,2\pi) \subset L^1(0,2\pi)$$

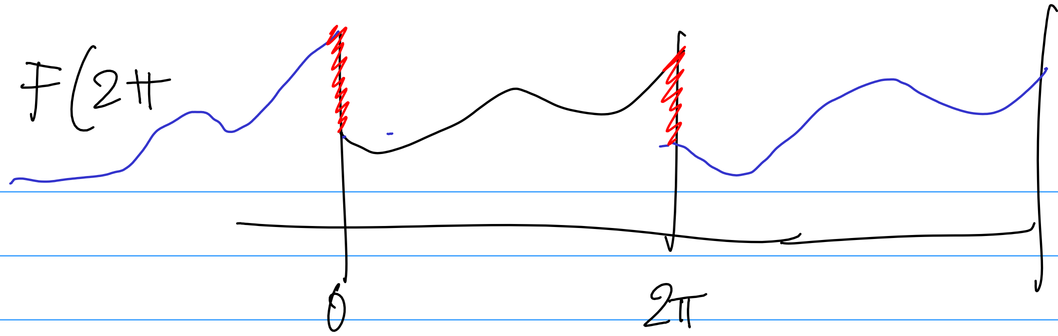
$$F(x) = c + \int_0^x f(y)\,dy \qquad F \in W^{1,2}(0,2\pi)$$

F è continua [assolutamente continua]

$$F(0) = c \qquad F(2\pi) = c + \int_0^{2\pi} f(x)\,dx$$

$$F \longrightarrow a_k \, b_k \qquad \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kx) + b_k \sin kx \right)$$

$F(0) \neq F(2\pi)$



$f \in L^2(0, 2\pi)$ tale che $\int_0^{2\pi} f(x)\,dx = 0$

$F(x) = c + \int_0^x f(y)\,dy$