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Some lecture notes on Integral Transforms

January 14, 2011

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Contents

1 Fourier Series . . . 5
1.1 Trigonometric Series . . . 5
1.2 Trigonometric Series in L^2 . . . 10
1.3 Some sufficient conditions for the convergence of a Fourier series . . 14
1.4 The Riemann-Lebesgue Lemma and its applications . . . 22

2 Fourier Transform . . . 27
2.1 Generalities . . . 27
2.2 The Schwartz Space . . . 31
2.3 The dual space . . . 36
2.4 Heisenberg Principle . . . 41
2.5 Bernoulli numbers . . . 44
2.6 Some applications to PDE . . . 50
2.6.1 The solution of the heat equation . . . 50
2.6.2 The solution of the Laplace equation . . . 51

3 Laplace Transform . . . 57
3.1 Recalls on Banach spaces . . . 57
3.2 Laplace Transform . . . 59
3.2.1 An application: the heat equation . . . 63
3.2.2 An application: the Abel equation . . . 64
3.2.3 Laplace transform and ODE . . . 70
3.3 Inverse Laplace transform . . . 73

A Some integral computations . . . 77

B Some useful Fourier series and integrals . . . 83

Bibliography . . . 85

References . . . 85

Chapter 1
Fourier Series

1.1 Trigonometric Series

Definition 1. Let $\{a_0, a_1, b_1, a_2, b_2, \dots\}$, $a_i, b_i \in \mathbb{R}$, be a set of numbers. Something like

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$$

is said to be a *trigonometric series*.

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We want to study the convergence of this series, in other words, if we define the finite sum

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)],$$

we want to study when there exists a function $s(x)$ such that $s_n(x) \to s(x)$. In this case we write

$$s(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)].$$

Let us suppose that

$$\frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|) < \infty$$

then

$$\left| \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] \right| \leq \frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|) < \infty$$

hence the trigonometric series exists and $s_n(x) \to s(x)$ uniformly. Since for each $n \in \mathbb{N}$ $s_n(x)$ is continuous and periodic of period 2π, so is $s(x)$.

Hence

$$\frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|) < \infty$$

is a first sufficient condition for the convergence of the trigonometric series.

Let us recall some properties of the trigonometric function sin and cos. Integrating by parts we have:

$$\int_0^{2\pi} \cos(kx)dx = \begin{cases} 2\pi \text{ if } k = 0 \\ 0 \text{ if } k \neq 0 \end{cases}$$

$$\int_0^{2\pi} \cos(nx)\cos(mx)dx = \begin{cases} 2\pi \text{ if } n = m = 0 \\ \pi \text{ if } n = m, n \neq 0 \\ 0 \text{ if } n \neq m \end{cases}$$

$$\int_0^{2\pi} \sin(nx)\sin(mx)dx = \begin{cases} \pi \text{ if } n = m, n > 0 \\ 0 \text{ if } n \neq m \end{cases}$$

$$\int_0^{2\pi} \sin(nx)\cos(mx)dx = 0$$

As we have just proved, if

$$\frac{|a_0|}{2} + \sum_{k=1}^{\infty} (|a_k| + |b_k|) < \infty,$$

$s_n(x) \to s(x)$ uniformly and $s(x)$ is continuous and 2π-periodic, hence we can integrate it. Using the properties above we can calculate the coefficens of the series and we obtain that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} s(x)\cos(nx)dx \quad \forall\, n \geq 0$$

and

$$b_n = \frac{1}{\pi} \int_0^{2\pi} s(x)\sin(nx)dx \quad \forall\, n \geq 1,$$

where a_n and b_n are called *Fourier coefficients* for $s(x)$.

Let us introduce some notations. The vector space of all sequences in $\mathbb{R}$ is denoted by $\mathbb{R}^\infty$.

$$l_p = \left\{ \alpha \in \mathbb{R}^\infty : \sum |\alpha_k|^p < \infty \right\} \quad p \geq 1$$

is a vector space on $\mathbb{R}$ and a Banach space with respect to the norm $\|\alpha\|_p = (\sum |\alpha_k|^p)^{1/p}$.

$$l_\infty = \{ \alpha \in \mathbb{R}^\infty : |\alpha_k| \leq C < \infty \}$$

is a vector space on $\mathbb{R}$ and a Banach space with respect to the norm $\|\alpha\|_\infty = \sup_k |\alpha_k|$.

Remark 1. $l_p \subset l_{p'}$ if $p < p' \quad \Rightarrow \quad l_1 \subset l_p \subset \ldots \subset l_{p'} \subset l_\infty$.

Let $\alpha = \{a_0, a_1, b_1, a_2, b_2, \ldots\} \in \mathbb{R}^\infty$. We can associate to α the partial sum $s_n(x)$ for all $n \geq 1$. Hence we associate to α a new sequence $\{s_0(x), s_1(x), s_2(x), \ldots\}$ where $s_n(x) \in C^\infty$ 2π-periodic for all n.

Theorem 1. *If $\alpha \in l_1$ then $s_n(x) \to s(x)$ uniformly. Moreover we can compute the Fourier coefficients a_k, b_k of $s(x)$.*

In general, a way to choose the sequence α is to take a function $f \in L^1(0,2\pi)$ and then compute

$$a_n(f) = \int_0^{2\pi} f(x)\cos(nx)dx \quad \text{and} \quad b_n(f) = \int_0^{2\pi} f(x)\sin(nx)dx.$$

Example 1. We want to prove that

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \varphi(x) \quad \text{in } [0,2\pi]$$

where $\varphi(x)$ is a suitable parabola.

First of all we can consider a parabola $p(x) = a(x-\pi)^2 + c$. Let us observe that

$$\varphi(0) = \varphi(2\pi) = \sum_{k=1}^{\infty} \frac{1}{k^2} \quad \text{and} \quad \varphi(\pi) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}.$$

If we put

$$\sigma = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

it is easy to proof that

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -\frac{\sigma}{2}.$$

Hence, after some computation, we have that

$$p(x) = \frac{3\sigma}{2\pi^2}(x-\pi)^2 - \frac{\sigma}{2}.$$

Let us compute the Fourier coefficients for $p(x)$. We have that $a_0 = 0$, $a_n = \frac{1}{n^2}$ and $b_n = 0$ for all $n \geq 1$. Now let us consider

$$s(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}.$$

It is trivial to show that $a_0 = 0$ and $b_n = 0$ for all $n \geq 1$. After some computation we also have that $a_n = \frac{1}{n^2}$ for all $n \geq 1$. Hence $s(x)$ and $p(x)$ have the same Fourier coefficient. Is it suffiecient to state that $s(x) = p(x)$ in $[0,2\pi]$?

The answer of this question is given by the following theorem:

Theorem 2. *Let $f \in C^0(0,2\pi)$ such that $a_n(f) = b_n(f) = 0$ for all n. Then $f \equiv 0$.*

Now we can use this theorem to complete the example above and state that

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} = \frac{3\sigma}{2\pi^2}(x-\pi)^2 - \frac{\sigma}{2} \quad \text{in } [0, 2\pi].$$

To prove the theorem above we use the following lemma:

Lemma 1.

$$\frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] = P(\cos(x), \sin(x)),$$

where $P(\xi, \eta)$ is a polynomial in two variables of degree n. Vice versa any trigonometric polynomial of degree n can be written as above with suitable coefficients.

Proof. From the identity $\exp(ix) = \cos(x) + i\sin(x)$ we get

$$\begin{aligned} \cos(nx) + i\sin(nx) &= \exp(inx) = (\cos(x) + i\sin(x))^n \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} \cos^{n-2k}(x)\sin^{2k}(x) \\ &\quad + i \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(x)\sin^{2k+1}(x) \end{aligned}$$

from which we have

$$\cos(nx) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n}{2k} \cos^{n-2k}(x)\sin^{2k}(x)$$

$$\sin(nx) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{2k+1} \cos^{n-2k-1}(x)\sin^{2k+1}(x).$$

Hence we get that

$$\frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] = P(\cos(x), \sin(x)), \tag{1.1}$$

where $P(\xi, \eta)$ is a polynomial in two unknowns of degree n.

On the other hand any trigonometric polynomial of degree n can be written as (1.1) with suitable coefficients. It is sufficient to prove it for

$$\cos^{n-k}(x)\sin^k(x).$$

First, let us suppose k even, i.e., $k = 2m$: we can write

$$\cos^{n-k}(x)\sin^k(x) = \cos^{n-k}(x)(1-\cos^2(x))^m$$

and this observation allows to prove the assertion only for $\cos^m(x)$; this can be seen by induction and using the identity $2\cos(kx)\cos(x) = \cos((k+1)x) + \cos((k-1)x)$. Further, we remark that in the sum (1.1) all terms $b_k = 0$.
If k is odd, i.e., $k = 2m+1$, we can write, exploiting the preceding remark

$$\cos^{n-k}(x)\sin^k(x) = \left(\frac{a_0}{2} + \sum_{j=1}^{n-1} a_j \cos(jx)\right)\sin(x),$$

where we use the identity

$$2\cos(jx)\sin(x) = \sin((j+1)x) - \sin((j-1)x)$$

to finish the proof and observing that in this case all $a_k = 0$ in (1.1).

Proof (Proof of the theorem 2). **Step 1.** Any trigonometric polynomial $P(\cos(x), \sin(x))$ of degree n, where $P(\xi, \eta)$ is a polynomial in two unknown with degree n, can be written as

$$P(\cos(x), \sin(x)) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)]$$

with suitable $a_0, a_1, b_1, \dots, a_n, b_n$. Hence we have that for any trigonometric polynomial it holds

$$\int_0^{2\pi} f(x)P(\cos(x), \sin(x))dx = 0.$$

Step 2. Let us suppose that f is not identically zero and without loss of generality we can suppose there exists a point x_0 where the function is positive; that implies that there exists an interval $[x_0 - \delta, x_0 + \delta]$ centred in x_0 where the function $f(x) \geq c > 0$ for some c.
Let us consider now the function $p(x) = \cos(x - x_0) + 1 - \cos(\delta)$ with the property that, in the interval $[0, 2\pi]$, $p(x) > 1$ for $|x - x_0| < \delta$ and $|p(x)| \leq 1$ for $|x - x_0| \geq \delta$. $p(x)^n$ is a trigonometric polynomial, hence

$$0 = \int_0^{2\pi} f(x)p(x)^n dx = \int_{|x-x_0|\leq\delta} f(x)p(x)^n dx + \int_{[0,2\pi]\setminus(x_0-\delta,x_0+\delta)} f(x)p(x)^n dx.$$

Now, with $\delta' < \delta$

$$\int_{|x-x_0|\leq\delta} f(x)p(x)^n dx \geq \int_{|x-x_0|\leq\delta'} cp(x_0+\delta')^n dx = 2\delta' cp(x_0+\delta')^n,$$

from which we have that the integral tends to infinite as n tends to infinite, because $p(x_0 + \delta') > 1$. On the other hand,

$$\left|\int_{[0,2\pi]\setminus(x_0-\delta,x_0+\delta)} f(x)p(x)^n dx\right| \leq 2\pi \max_{0\leq x\leq 2\pi} |f(x)|.$$

Hence we have $A_n + B_n = 0$ where $A_n \to \infty$ and B_n is bounded: that is not possible; hence f must be identically zero.

Let us consider now a function f. If

$$f \in L^1(0,2\pi) := \left\{ f : [0,2\pi] \to \mathbb{R} : \text{ f is measurable and } \int_0^{2\pi} |f(x)|\,dx < +\infty \right\},$$

we can compute the coefficients

$$a_n(f) = \int_0^{2\pi} f(x)\cos(nx)dx \quad \text{and} \quad b_n(f) = \int_0^{2\pi} f(x)\sin(nx)dx.$$

Which criterion have we got for the convergence of the numeric series

$$\alpha = (a_0, a_1, b_1, a_2, b_2, a_3, \ldots)?$$

In 1924 Kolmogorov gave an example for the non convergence of the series:

Theorem 3 (Kolmogorov). *There exists a function $f \in L^1(0,2\pi)$ such that the corresponding sequence s_n doesn't converge in any point.*

However, if we restrict the choice of f, we have a sufficient condition for the convergence of α.

Theorem 4. *Let $f \in L^2(0,2\pi)$ then $s_n(x) \to f(x)$ in the $L^2(0,2\pi)-$sense, that is*

$$\int_0^{2\pi} |s_n(x) - f(x)|^2\,dx \longrightarrow 0.$$

Remark 2. This theorem says in particular that

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k\cos(kx) + b_k\sin(kx)) \text{ in } L^2\text{-sense}$$

The theorem states that, if we take $f \in L^2(0,2\pi)$, $s_n(x)$ converges to $f(x)$ in L^2-sense. Lusin conjectured that this convergence had to be a convergence almost everywhere. Lusin's conjecture was proved by Carleson in 1966 and in 1967 Hunt generalized the theorem for $f \in L^p(0,2\pi)$ for all $p > 1$ (for $p = 1$ it doesn't hold, as the example of Kolmogorov shows).
For the proof of the theorem 4, we need some general facts about the Hilbert spaces.

1.2 Trigonometric Series in L^2

Let H be a separable Hilbert space.

Definition 2. A collection $\{e_1, e_2, e_3, \ldots\} \subset H$ such that $\langle e_i, e_j \rangle = 0$ for every $i \neq j$ is called *orthogonal system* of vectors. If in addition $\langle e_i, e_i \rangle = \|e_i\|^2 = 1\ \forall i \in \mathbb{N}$, it is called *orthonormal system*.

To each $\alpha \in l^2(\mathbb{N})$ we can associate a sequence of vectors $\{v_n\}$, where $v_n := \sum_{i=1}^n \alpha_i e_i$. Then, for the completeness of H, $v_n \to v$ in H. If we define

$$V_0 := \left\{ \sum_{i=1}^n \alpha_i e_i : \ \alpha \in l_2, n \in \mathbb{N} \right\}$$

in particular $\{v_n\} \subset V_0$ and, since V_0 is a subspace of H, $V := \bar{V_0}$ is complete and $v \in V$.
On the other hand we have the following theorem:

Theorem 5. *Let H be a Hilbert space, $v \in H$ and $\alpha = \{\alpha_n\}$ be the sequence defined by $\alpha_n = \langle v, e_n \rangle$. Then $\alpha \in l_2(\mathbb{N})$.*

Proof. Let $v_n := \sum_{i=1}^n \langle v, e_i \rangle e_i$.
Then $\|v_n\|^2 = \sum_{i=1}^n \langle v, e_i \rangle^2$, but also $\langle v_n, v \rangle = \sum_{i=1}^n \langle v, e_i \rangle^2$, that is

$$\|v_n\|^2 = \langle v_n, v \rangle = \sum_{i=1}^n \alpha_n^2. \tag{1.2}$$

Notice that $v = v_n + (v - v_n)$ and v_n and $v - v_n$ are orthogonal, indeed $\langle v_n, v - v_n \rangle = \langle v_n, v \rangle - \langle v_n, v_n \rangle = 0$. Then

$$\|v_n\| \leq \|v\| < \infty. \tag{1.3}$$

From (1.2) and (1.3), follows that $\alpha = \{\alpha_n\} \in l^2(\mathbb{N})$.

Proof (of Theorem 4). Recall that $L^2(0, 2\pi)$ with the scalar product

$$\langle f, g \rangle = \int_0^{2\pi} f(x)g(x)dx \quad \forall f, g \in L^2(0, 2\pi)$$

is a Hilbert space and define

$$u_0 := \frac{1}{\sqrt{2\pi}} \quad , \quad u_n := \frac{\cos(nx)}{\sqrt{\pi}} \quad \text{and} \quad v_n := \frac{\sin(nx)}{\sqrt{\pi}} \quad \forall n \geq 0$$

Notice that

- $\|u_0\| = \|u_n\| = \|v_n\| = 1 \ \forall n \geq 1$
- $\langle u_n, u_m \rangle = 0 \ \ \forall n \neq m$
- $\langle v_n, v_m \rangle = 0 \ \ \forall n \neq m$
- $\langle u_n, v_m \rangle = 0 \ \ \forall n, m.$

Then $\{u_0, u_1, v_1, u_2, v_2, ...\}$ is an orthonormal system in $L^2(0, 2\pi)$. It is called *Fourier system*.
Let $f \in L^2(0, 2\pi)$, and, using the functions just defined, notice that

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)dx = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} u_n(x) f(x)dx = \frac{1}{\sqrt{\pi}} \langle f, u_n \rangle$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} v_n(x)f(x)dx = \frac{1}{\sqrt{\pi}}\langle f, v_n\rangle$$

for $n \geq 1$ and

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \sqrt{\frac{2}{\pi}}\int_0^{2\pi} u_0(x)f(x)dx = \sqrt{\frac{2}{\pi}}\langle f, u_0\rangle .$$

Then it is not hard to see that

$$\frac{a_0}{2} = \langle f, u_0\rangle u_0(x) \qquad a_k\cos(kx) = \langle f, u_k\rangle u_k(x) \qquad b_k\sin(kx) = \langle f, v_k\rangle v_k(x)$$

hence

$$s_n(x) = \langle f, u_0\rangle u_0(x) + \sum_{k=1}^{n}\left(\langle f, u_k\rangle u_k(x) + \langle f, v_k\rangle v_k(x)\right)$$

Applying the previous theorem with $H = L^2$, $v = f$ and the orthonormal system given by the Fourier system, it follows that

$$s_n \text{ converges to an } s(x) \in L^2(0, 2\pi). \tag{1.4}$$

It remains to prove that $s(x) = f(x)$ and it is a consequence of the following theorem and the fact that $L^2(0,2\pi) \subset L^1(0,2\pi)$.

Theorem 6. *Let $f \in L^1(0,2\pi)$ and let $a_n = b_n = 0$ for every $n \geq 0$. Then $f = 0$ almost everywhere.*

Proof. Define

$$F(x) := \int_0^x f(y)dy.$$

Then F is continuous and derivable almost everywhere.
Since $a_n = b_n = 0 \;\; \forall n$, it is not hard to show that also the Fourier coefficients of F, A_n, B_n, are 0, except at most A_0. This means that the Fourier coefficients of $F - \frac{A_0}{2}$ are all 0.
Since $F - \frac{A_0}{2}$ is continuous, we can now apply theorem 2 to show that $F(x) = \frac{A_0}{2}$ for all $x \in [0, 2\pi]$. In particular $0 = F(0) = \frac{A_0}{2}$.
Hence F is continuous and all its coefficients are 0. Applying again theorem 2, it follows that

$$F(x) = 0 \;\; \forall x$$

and

$$F^{'}(x) = f(x) = 0 \text{ for almost every } x \in [0, 2\pi].$$

To prove statement (1.4), there is also a more directly way:

Theorem 7. *If $s_n(x) \in L^2(0,2\pi)$, $s_n(x) \to s(x)$ in L^2.*

Proof. It is sufficient to prove that $(s_n(x))_n$ is a Cauchy sequence, i.e. $\left\|s_{n+p}(x) - s_n(x)\right\|_2 \to 0$ for all p, if $n \to \infty$. If it is true, $\exists\ s(x) \in L^2(0,2\pi)$ such that $s_n(x) \to s(x)$ in L^2 because of the completeness of L^2. We have that

$$s_{n+p}(x) - s_n(x) = \sum_{k=n+1}^{n+p} \left(\langle u_k, f \rangle u_k(x) + \langle v_k, f \rangle v_k(x) \right).$$

After some computations, using the definition of orthonormal system, we have that

$$\left\| s_{n+p} - s_n \right\|_2^2 = \int_0^{2\pi} \left(s_{n+p}(x) - s_n(x) \right)^2 dx = \sum_{k=n+1}^{n+p} \left(\langle u_k, f \rangle^2 + \langle v_k, f \rangle^2 \right).$$

Using the Cauchy-Schwartz inequality, we obtain that

$$\|s_n\|_2^2 = \langle s_n, f \rangle \leq \|s_n\|_s \|f\|_2 \quad \Rightarrow \quad \|s_n\|_2^2 \leq \|f\|_2^2.$$

Hence

$$\langle u_0, f \rangle^2 + \sum_{k=1}^{n} \left(\langle u_k, f \rangle^2 + \langle v_k, f \rangle^2 \right) \leq \|f\|_2^2 \quad \text{(Bessel Inequality)}$$

Hence

$$\sum_{k=1}^{n} \left(\langle u_k, f \rangle^2 + \langle v_k, f \rangle^2 \right) < +\infty \quad \Rightarrow \quad \left\| s_{n+p}(x) - s_n(x) \right\|_2^2 \to 0.$$

Example 2. We want to apply this theory to the function

$$f(x) = \frac{\pi - x}{2} \quad \text{for } 0 \leq x \leq 2\pi.$$

Then $f \in L^2(0, 2\pi)$ and $a_n = 0 \ \forall n$ and $b_k = \frac{1}{k}$, that is

$$\alpha = \{0, 0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, ...\}$$

Then $\alpha \in l_2$, but $\alpha \notin l_1$ and we can't apply theorem 1, so we won't have a uniform convergence of $s_n(x)$.
However, since $f \in L^2(0, 2\pi)$,

$$\frac{\pi - x}{2} = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k} \quad \text{in the } L^2\text{−sense.}$$

From this result, we can also compute a particular series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{\pi} \int_o^{2\pi} f(x)^2 dx = \frac{\pi^2}{6}.$$

1.3 Some sufficient conditions for the convergence of a Fourier series

Until now, we have shown that a Fourier sequence with coefficients in l_1 converges uniformly to a function in $C(0,2\pi)$, while there is a bijection between the sequences with coefficients in l_2 and the functions in $L^2(0,2\pi)$, but in this case we have only a L^2-convergence and we loose the uniform convergence.
Our goal is to extend these relationships. We will see that the following diagram holds

$$\begin{array}{ccccc} l_1 & \subset & l_2 & \subset & c_0 \\ \downarrow & & \updownarrow & & \uparrow \\ C(0,2\pi) & \subset & L^2(0,2\pi) & \subset & L^1(0,2\pi) \end{array}$$

where the downarrows mean that the Fourier sequence of the coefficients converges to a function in the pointed space, while the uparrows mean that the Fourier coefficents of the considered function are in the pointed space.
We have already proved all the implications, except the last one that is the content of the Riemann Lebesgue theorem. We will consider it in the next section.
For the vice versa of these arrows there exist only sufficient conditions and we propose us to study some of them.

Theorem 8. *Let us suppose that we can write $f \in C^0(0,2\pi)$ as $f(x) = c + \int_0^x g(\xi)d\xi$ with*

(i) $g \in L^2(0,2\pi)$
(ii) $\int_0^{2\pi} g(\xi)d\xi = 0$

then

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos(kx) + b_k\sin(kx)) \quad \textit{uniformly in } [0,2\pi].$$

Remark 3. Notice that the theorem states only the uniform convergence of the Fourier series. It is not implied that the coefficients are in l_1.

Remark 4. Condition (ii) of the theorem allows us to extend the function f periodically on $\mathbb{R}$ in a continuous way.
To study such a continuous periodic function we can consider every inteval $[c, c+2\pi]$. Then in some cases it can be useful to consider the function f defined on the circle, T_1.
In this notation, assumption (ii) of the theorem requires that $f \in C^0(T_1)$.

Proof. Since $g \in L^2(0,2\pi)$, we can compute the Fourier coefficients of g:

$$\alpha_n = \frac{1}{\pi}\int_0^{2\pi} g(x)\cos(nx)dx \qquad \beta_n = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin(nx)dx.$$

Remember that $g(x) = f'(x)$ for a. e. x, then, integrating by parts,

$$\alpha_n = nb_n \qquad \text{and} \qquad \beta_n = -na_n \tag{1.5}$$

for all $n \geq 1$, where a_n and b_n are the Fourier coefficients of f. We have to show that the partial sum $s_n(x)$ converges uniformly to $f(x)$. To reach this goal we show that $\{s_n\}$ is a Cauchy sequence. Indeed

$$\begin{aligned}
\left|s_{n+p}(x) - s_n(x)\right| &= \left|\sum_{k=n+1}^{n+p} (a_k \cos(kx) + b_k \sin(kx))\right| \\
&= \left|\sum_{k=n+1}^{n+p} \sqrt{a_k^2+b_k^2}\left(\frac{a_k}{\sqrt{a_k^2+b_k^2}}\cos(kx) + \frac{b_k}{\sqrt{a_k^2+b_k^2}}\sin(kx)\right)\right| \\
&= \left|\sum_{k=n+1}^{n+p}\left(\sqrt{a_k^2+b_k^2}\,\cos(kx-\varphi_k)\right)\right| \\
&\leq \sum_{k=n+1}^{n+p}\sqrt{a_k^2+b_k^2} = \sum_{k=n+1}^{n+p}\frac{\sqrt{\alpha_k^2+\beta_k^2}}{k} \\
&\leq \left(\sum_{k=n+1}^{n+p}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k=n+1}^{n+p}(\alpha_k^2+\beta_k^2)\right)^{\frac{1}{2}}
\end{aligned}$$

where, φ_k is such that $\cos(\varphi_k) = \frac{a_k}{\sqrt{a_k^2+b_k^2}}$, $\sin(\varphi_k) = \frac{b_k}{\sqrt{a_k^2+b_k^2}}$, and we have used result (1.5) and, in the last line, the Cauchy-Schwarz inequality. Notice that the first factor tends to zero when n goes to infinity, because

$$\sum_{k=1}^{\infty}\frac{1}{k^2} < \infty,$$

while the second factor tends to zero, because

$$\sum_{k=1}^{\infty}(\alpha_k^2+\beta_k^2) \leq \frac{1}{\pi}\int_0^{2\pi} g(x)^2 dx$$

that is bounded. This means that $\{s_n\}$ is a Cauchy sequence, hence it converges to an $\tilde{f}(x) \in C^0(0,2\pi)$. Since f and $\tilde{f}$ have the same Fourier coefficients, from theorem 2 follows that $f \equiv \tilde{f}$.

We would like to extend this condition also for a function $f \in C^0(0,2\pi)$ that doesn't satisfy condition (ii), that is that can't be extended periodically to a continuous function on $\mathbb{R}$. Some examples on the computer, make us suppose that the Fourier series of such a function converges uniformly to f, but not in the whole interval $[0,2\pi]$, but in a restricted one, as $[\delta, 2\pi-\delta]$, with $\delta > 0$.

To prove it we will make use of the function

$$h(x) = \frac{\pi - x}{2},$$

that we can write also as

$$h(x) = \frac{\pi}{2} + \int_0^x -\frac{1}{2} d\xi.$$

Its Fourier coeffincients are $a_k = 0$ and $b_k = \frac{1}{k}$, so the sequence $\{a_0, a_1, b_1, a_2, ...\} \in l^2$. Then

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$$

converges to an $\tilde{h}(x)$ in L^2.

Consider $f \in C^0(0, 2\pi)$ such that

$$f(x) = c + \int_0^x g(\xi) d\xi,$$

with $g \in L^2(0, 2\pi)$, and $f(0) \neq f(2\pi)$, and extend it periodically on $\mathbb{R}$. This function is discontinuous in all the points $0 + 2k\pi$, $k \in \mathbb{Z}$. Let us introduce the size of the discontinuity $size(f) = f(0^+) - f(0^-)$. If we consider $h(x)$ we have that $size(h) = \pi$. Then the function

$$\widetilde{f}(x) = f(x) - ch(x) \quad \text{where} \quad c = \frac{f(0^+) - f(0^-)}{\pi}$$

is continuous and we can apply the previous theorem. By linearity, the partial sum is

$$\widetilde{s_n}(x) = s_n(x) - ch_n(x) \quad \Rightarrow \quad s_n(x) = \widetilde{s_n}(x) + ch_n(x).$$

Using the theorem we have that $\widetilde{s_n}(x) \to \widetilde{f}(x)$ uniformly. If we prove that $h_n(x) \to h(x)$ uniformly in $[\delta, 2\pi - \delta]$, $\delta > 0$, it follows that $s_n(x) \to f(x)$ uniformly in $[\delta, 2\pi - \delta]$. Hence we have proved the theorem without condition (ii).

It remains to prove that $h_n(x) \to h(x)$ uniformly in $[\delta, 2\pi - \delta]$, $\delta > 0$.

Proof. Let us consider

$$\hat{h}(x) = (1 - \cos(x))h(x).$$

After some computations we obtain that $\hat{a}_n = 0$ for all $n \geq 0$, $\hat{b}_1 = b_1 - \frac{1}{2}b_2 = \frac{3}{4}$, and

$$\hat{b}_n = b_n - \frac{1}{2}(b_{n+1} + b_{n-1}) = -\frac{1}{n}\frac{1}{n^2 - 1} \quad \forall\, n \geq 2.$$

Moreover $\hat{h}(x)$ satisfies conditions (i) and (ii) of theorem 8, hence

$$\hat{h}_n(x) = \frac{3}{4}\sin(x) + \sum_{k=2}^{n} \frac{\sin(kx)}{k(k^2 - 1)} \longrightarrow \hat{h}(x) \quad \text{uniformly.}$$

Now

$$(1 - \cos(x))h_n(x) = \hat{h}_n(x) - \frac{1}{2}b_n \sin((n+1)x) + \frac{1}{2}b_{n+1}\sin(nx),$$

where $\hat{h}_n(x) \to \hat{h}(x)$ uniformly and $\frac{1}{2}b_n \sin((n+1)x)$, $\frac{1}{2}b_{n+1}\sin(nx) \to 0$ uniformly, because $b_n \to 0$ and $\sin(\cdot)$ is bounded.

Hence $(1-\cos(x))h_n(x) \to \hat{h}(x)$ uniformly, that is

$$(1-\cos(x))(h_n(x)-h(x)) \to 0 \quad \text{uniformly}.$$

Since $1-\cos(x) \geq 1-\cos(\delta)$, for each $x \in [\delta, 2\pi-\delta]$, with $\delta > 0$, it follows that

$$h_n(x) \to h(x) \quad \text{uniformly in } [\delta, 2\pi-\delta],\ \delta > 0.$$

We are now interested in the Fourier sereies of the function

$$f(x) = \cos(\alpha x) \in L^1(0,2\pi), \text{ where } \alpha \notin \mathbb{Z}.$$

The period of this function is $\frac{2\pi}{\alpha}$, then its Fourier series is not the function itself. We would like to find its Fourier series and to study its convergence.

More generally, let us consider $f \in L^1(0,2\pi)$ and its Fourier coefficients a_n and b_n. The partial sums $s_n(x)$ exist for all $n \geq 0$, but, as the theorem of Kolmogorov shows, we don't know anything about their limit. However, we can always compute

$$\begin{aligned} s_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)] \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(y)dy + \frac{1}{\pi}\int_0^{2\pi} f(y)\sum_{k=1}^{n}[\cos(ky)\cos(kx)+\sin(ky)\sin(kx)]dy \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(y)dy + \frac{1}{\pi}\int_0^{2\pi} f(y)\sum_{k=1}^{n}\cos(k(x-y))dy \\ &= \frac{1}{\pi}\int_0^{2\pi} f(y)[\frac{1}{2}+\sum_{k=1}^{n}\cos(k(x-y))]dy \end{aligned}$$

Definition 3. $D_n(\vartheta) := \frac{1}{\pi}\left[\frac{1}{2}+\sum_{k=1}^{n}\cos(k\vartheta)\right]$ is called *Dirichlet kernel*.

Hence

$$s_n(x) = \int_0^{2\pi} f(y)D_n(x-y)dy.$$

Proposition 1 (Properties of Dirichlet kernel). *The following statements hold:*

(i) $\int_0^{2\pi} D_n(\vartheta)d\vartheta = 1$

(ii) $D_0(\vartheta) = \frac{1}{2\pi}$

(iii) $D_n(\vartheta) = \frac{\sin(\frac{2n+1}{2}\vartheta)}{2\pi\sin(\frac{\vartheta}{2})}$ *for all* $n \geq 1$

Proof. The proof of the statements (i) and (ii) is trivial. Let us prove (iii).

$$\frac{1}{2}+\sum_{k=1}^{n}\cos(kx) = \Re[1+\exp(i\vartheta)+\exp(2i\vartheta)+\ldots+\exp(ni\vartheta)] - \frac{1}{2}$$

Let $z = \exp(i\vartheta)$. Then $1+z+z^2+\ldots+z^n = \frac{z^{n+1}-1}{z-1}$. Hence

$$\Re\left[\frac{z^{n+1}-1}{z-1}\right] = \Re\left[\frac{(\overline{z-1})(z^{n+1}-1)}{(\overline{z-1})(z-1)}\right] = \frac{\Re\left[(\bar{z}-1)(z^{n+1}-1)\right]}{(\bar{z}-1)(z-1)}$$

Now

$$(\bar{z}-1)(z-1) = z\bar{z}-z-\bar{z}+1 = 2-(z+\bar{z}) = 2(1-\frac{z+\bar{z}}{2}) = 2(1-\cos(\vartheta))$$

$$(\bar{z}-1)(z^{n+1}-1) = \bar{z}zz^n - z^{n+1} - \bar{z} + 1$$

Hence

$$\Re\left[\frac{z^{n+1}-1}{z-1}\right] = \frac{\cos(n\vartheta)-\cos((n+1)\vartheta)-\cos(\vartheta)+1}{2(1-\cos(\vartheta))}$$

and

$$\frac{1}{2}+\sum_{k=1}^{n}\cos(kx) = \frac{\cos(n\vartheta)-\cos((n+1)\vartheta)-\cos(\vartheta)+1}{2(1-\cos(\vartheta))} - \frac{1}{2} = \frac{\cos(n\vartheta)-\cos((n+1)\vartheta)}{2(1-\cos(\vartheta))}$$

Now

$$\cos(n\vartheta) = \cos((n+\frac{1}{2})\vartheta-\frac{\vartheta}{2}) = \cos((n+\frac{1}{2})\vartheta)\cos(\frac{\vartheta}{2})+\sin((n+\frac{1}{2})\vartheta)\sin(\frac{\vartheta}{2}) \tag{1.6}$$

$$\cos((n+1)\vartheta) = \cos((n+\frac{1}{2})\vartheta+\frac{\vartheta}{2}) = \cos((n+\frac{1}{2})\vartheta)\cos(\frac{\vartheta}{2})-\sin((n+\frac{1}{2})\vartheta)\sin(\frac{\vartheta}{2}) \tag{1.7}$$

$$1-\cos(\vartheta) = 2\sin^2(\frac{\vartheta}{2}) \tag{1.8}$$

Hence, using (1.6), (1.7) and (1.8), we obtain (iii).

Let us consider $f \in L^1(0,2\pi)$. We have seen that $s_n(x) = \int_0^{2\pi} f(y)D_n(x-y)dy$. What can we say about the convergence of this series?

We can't hope to obtain a convergence in the usual meaning. So we want to introduce another kind of convergence: the convergence in the Cesaro sense.
Let $(a_n)_n$ be a sequence such that $a_n \to a$. We can introduce a new sequence

$$a_1,\ \frac{a_1+a_2}{2},\ \frac{a_1+a_2+a_3}{3},\ \ldots \quad \text{i.e. } (\sigma_n)_n \text{ where } \sigma_n = \frac{a_1+\ldots+a_n}{n}$$

We can prove that, if $a_n \to a$, then also $\sigma_n \to a$, while the contrary isn't true in general.
This convergence is called *convergence in the Cesaro sense*, from the name of Ernesto Cesàro, the mathematician, who introduced it, and extends the set of sequences that are convergent in the usual meaning.

We want to study the Cesaro convergence of $(s_n(x))_n$. Let

$$\sigma_n(x) = \frac{s_0(x) + s_1(x) + \ldots + s_{n-1}(x)}{n}$$

be the Cesaro sum of $(s_n(x))_n$, then

$$\sigma_n(x) = \int_0^{2\pi} f(y)(\frac{D_0 + D_1 + \ldots + D_{n-1}}{n})(x-y)dy.$$

Definition 4. $K_n(\vartheta) := \frac{D_0(\vartheta)+D_1(\vartheta)+\ldots+D_{n-1}(\vartheta)}{n}$ is called *Feyer kernel*.

Fig. 1.1 Constantin Caratheodory (Berlin 1873, Munich 1950) and Leopold Fejér (Pécs 1880, Budapest 1959)

Hence

$$\sigma_n(x) = \int_0^{2\pi} f(y)K_n(x-y)dy.$$

It is easy to prove the following proposition:

Proposition 2 (Properties of Feyer kernel). *The following statements hold:*

(i) $\int_0^{2\pi} K_n(\vartheta)d\vartheta = 1$
(ii) $K_n(\vartheta) \geq 0$
(iii) $K_n(\vartheta) = \frac{1}{2\pi n}\frac{\sin^2(\frac{n\vartheta}{2})}{\sin^2(\frac{\vartheta}{2})}$ *for all* $n \geq 0$.

In the notation that we have introduced, we can give the following sufficient condition:

Theorem 9. *If* f *is periodic and continuous, then* $\sigma_n(x) \to f(x)$ *uniformly.*

Remark 5. We stress that the Fourier series $\{s_n(x)\}_n$ can be divergent, while the theorem assures only the convergence in the Cesaro sense. However, if the Fourier series is convergent, then it converges also in the Cesaro sense.

Example 3. We want to compute the Fourier series of the funtions $f(x) = \cos(\alpha x)$, $g(x) = \sin(\alpha x)$ and $h(x) = \exp(\alpha x)$ with $\alpha \notin \mathbb{Z}$.
Let us observe first that

$$\cos(n\pi) = (-1)^n \;\; \forall\, n \in \mathbb{N} \quad \text{and} \quad \sin(n\pi) = 0 \;\forall\, n \in \mathbb{N} \tag{1.9}$$

Before to start let us recall the addition formulas for the function $\cos(x)$,

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \quad \cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y). \tag{1.10}$$

Let us start with $f(x)$.
We have trivially that $b_n = 0$ for each $n \in \mathbb{N}$, and that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\alpha x)dx = \left[\frac{\sin(\alpha x)}{\pi\alpha}\right]_{-\pi}^{\pi} = 2\frac{\sin(\alpha\pi)}{\pi\alpha}.$$

We have now to compute a_n with $n \geq 1$. Using (1.10) we have that

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(\alpha x)\cos(nx)dx \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos((\alpha+n)x) + \cos((\alpha-n)x))dx \\ &= \frac{1}{\pi}\left[\frac{(\alpha-n)\sin((\alpha+n)\pi) + (\alpha+n)\sin((\alpha-n)\pi)}{\alpha^2 - n^2}\right] \end{aligned}$$

Using (1.9) we have that

$$(\alpha-n)\sin((\alpha+n)\pi) + (\alpha+n)\sin((\alpha-n)\pi) = 2(-1)^n\alpha\sin(\alpha\pi). \tag{1.11}$$

Hence, using (1.11), we have that, for all $n \geq 1$,

$$a_n = \frac{1}{\pi}\frac{(-1^n)2\alpha\sin(\alpha\pi)}{\alpha^2 - n^2}.$$

Hence,

$$\boxed{\cos(\alpha x) = \frac{2\sin(\alpha\pi)}{\pi}\left(\frac{1}{2\alpha} + \sum_{n=1}^{\infty}(-1)^n\frac{\alpha}{\alpha^2 - n^2}\cos(nx)\right)}$$

Let us now consider the function $g(x)$.
First of all, let us observe that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(\alpha x)dx = 0,$$

because we are integrating an odd function on a simmetric interval. Moreover, trivially, $a_n = 0$ for each $n \geq 1$, hence $a_n = 0$ for all $n \in \mathbb{N}$. We have now to compute the b_n coefficients. Using (1.9) and (1.10), we have that

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(\alpha x)\sin(nx)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[\cos((\alpha-n)x) - \cos((\alpha+n)x)\right]dx \\
&= \frac{1}{2\pi}\left[\frac{(\alpha+n)\sin((\alpha-n)x) - (\alpha-n)\sin((\alpha+n)x)}{\alpha^2-n^2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\frac{(-1)^n 2n\sin(\alpha\pi)}{\alpha^2-n^2}.
\end{aligned}$$

Hence, we have

$$\boxed{\sin(\alpha x) = \frac{2\sin(\alpha\pi)}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{n}{\alpha^2-n^2}\sin(nx)}$$

Let us now consider the function $h(x)$.
First of all, let us observe that, integrating two times by parts, we obtain that

$$\int \exp(\alpha x)\cos(nx)dx = \frac{\alpha}{\alpha^2+n^2}\left(\exp(\alpha x)\cos(nx) + \frac{n}{\alpha}\exp(\alpha x)\sin(nx)\right), \quad (1.12)$$

and

$$\int \exp(\alpha x)\sin(nx)dx = \frac{\alpha}{\alpha^2+n^2}\left(\exp(\alpha x)\sin(nx) - \frac{n}{\alpha}\exp(\alpha x)\cos(nx)\right). \quad (1.13)$$

Before we start to compute the Fourier coefficients for $h(x)$, let us recall the definition of the hyperbolic functions $\cosh(x)$ and $\sinh(x)$,

$$\cosh(x) = \frac{\exp(x)+\exp(-x)}{2} \quad \text{and} \quad \sinh(x) = \frac{\exp(x)-\exp(-x)}{2}. \quad (1.14)$$

We have, trivially integrating and using (1.14), that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \exp(\alpha x)dx = \frac{2\sinh(\alpha\pi)}{\alpha\pi}.$$

Using (1.9), (1.12), (1.13) and (1.14) we have that

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \exp(\alpha x)\cos(nx)dx \\
&= \frac{\alpha}{\alpha^2+n^2}\frac{1}{\pi}\left[\exp(\alpha x)\cos(nx) + \frac{n}{\alpha}\exp(\alpha x)\sin(nx)\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\frac{\alpha}{\alpha^2+n^2}\left[\cos(n\pi)(\exp(\alpha\pi)-\exp(-\alpha\pi)) + \frac{n}{\alpha}\sin(n\pi)(\exp(\alpha\pi)+\exp(\alpha\pi))\right] \\
&= \frac{1}{\pi}\frac{\alpha}{\alpha^2+n^2}\left[2\cos(n\pi)\sinh(\alpha\pi) + 2\frac{n}{\alpha}\sin(n\pi)\cosh(\alpha\pi)\right] \\
&= \frac{1}{\pi}\frac{\alpha}{\alpha^2+n^2}(-1)^n 2\sinh(\alpha\pi),
\end{aligned}$$

for each $n \geq 1$, and, with similar computations,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \exp(\alpha x)\sin(nx)dx \\
&= \frac{1}{\pi}\frac{n}{\alpha^2+n^2}(-1)^{n+1}2\sinh(\alpha\pi),
\end{aligned}$$

for all $n \in \mathbb{N}$.
Hence,

$$\boxed{\exp(\alpha x) = \frac{2\sinh(\alpha\pi)}{\pi}\left[\frac{1}{2\alpha} + \sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha^2+n^2}(\alpha\cos(nx) - n\sin(nx))\right]}$$

Exercise 1 *Compute the Fourier series for the functions* $c(x) = \cosh(\alpha x)$ *and* $s(x) = \sinh(\alpha x)$ *with* $\alpha \notin \mathbb{Z}$.

1.4 The Riemann-Lebesgue Lemma and its applications

Let us recall the following diagram

$$\begin{array}{ccccc}
l_1 & \subset & l_2 & \subset & c_0 \\
\downarrow & & \updownarrow & & \uparrow \\
C(0,2\pi) & \subset & L^2(0,2\pi) & \subset & L^1(0,2\pi)
\end{array}$$

where $c_0 = \{\alpha \in \mathbb{R}^\infty : \ \alpha_n \to 0 \text{ when } n \to \infty\}$. The Riemann-Lebesgue lemma proves the uparrow $L^1(0,2\pi) \to c_0$.
Let us recall the following result:

Theorem 10 (Lebesgue). *If* $f \in L^1(a,b)$*, then*

$$\int_a^b |f(x+h) - f(x)|\, dx \to 0 \quad when \quad h \to 0.$$

Proof. See "S. Mizohata, *The theory of partial differential equations*, Cambridge University Press, London, 1973", Lemma 1.1.

Lemma 2 (Riemann-Lebesgue). *If* $f \in L^1(a,b)$,

$$\int_a^b f(x)\sin(nx)dx \to 0 \quad when \quad n \to \infty$$

$$\int_a^b f(x)\cos(nx)dx \to 0 \quad when \quad n \to \infty$$

Proof. We will consider the case of $\int_a^b f(x)\sin(nx)dx$. The proof for $\int_a^b f(x)\cos(nx)dx$ is analogous.

Let $n \in \mathbb{N}$ and consider the intervals $\left[\frac{2k\pi}{n}, \frac{2k+2}{n}\pi\right]$ contained in $[a,b]$. The integral of $\sin(nx)$ vanishes in all this intervals. Defining $A_{a,b} \subseteq \mathbb{Z}$ as the set of all $k \in \mathbb{Z}$ such that $\frac{2k\pi}{n}$ and $\frac{2k+2}{n}\pi$ are in $[a,b]$, we can consider the whole interval $[a,b]$ as the union over $k \in A_{a,b}$ of the intervals $\left[\frac{2k\pi}{n}, \frac{2k+2}{n}\pi\right]$ and the two smaller intervals, that have a resp. b as left resp. right border. Calling I_a and I_b the integral of $f(x)\sin(nx)$ calculated respectively on this two intervals, we obtain that

$$\int_a^b f(x)\sin(nx)dx = \sum_{k \in A_{a,b}} \int_{\frac{2k\pi}{n}}^{\frac{2k+2}{n}\pi} f(x)\sin(nx)dx + I_a + I_b$$

Let us consider

$$\int_{\frac{2k\pi}{n}}^{\frac{2k+2}{n}\pi} f(x)\sin(nx)dx = \int_{\frac{2k\pi}{n}}^{\frac{2k+1}{n}\pi} f(x)\sin(nx)dx + \int_{\frac{2k+1}{n}\pi}^{\frac{2k+2}{n}\pi} f(x)\sin(nx)dx$$

In the second integral make the substitution $x' = x - \pi/n$, then $\frac{2k\pi}{n} \le x' \le \frac{2k+1}{n}\pi$, because $\frac{2k+1}{n}\pi \le x \le \frac{2k+2}{n}\pi$, and

$$\begin{aligned}\int_{\frac{2k+1}{n}\pi}^{\frac{2k+2}{n}\pi} f(x)\sin(nx)dx &= \int_{\frac{2k\pi}{n}}^{\frac{2k+1}{n}\pi} f(x' + \frac{\pi}{n})\sin(n(x' + \frac{\pi}{n}))dx \\ &= -\int_{\frac{2k\pi}{n}}^{\frac{2k+1}{n}\pi} f(x' + \frac{\pi}{n})\sin(nx')dx\end{aligned}$$

where in the last equality we have used $\sin(n(x' + \pi/n)) = \sin(nx' + \pi) = \sin(nx')\cos(\pi) + \sin(\pi)\cos(nx') = -\sin(nx')$.
Hence

$$\int_{\frac{2k\pi}{n}}^{\frac{2k+2}{n}\pi} f(x)\sin(nx)dx = \int_{\frac{2k\pi}{n}}^{\frac{2k+1}{n}\pi} [f(x) - f(x+\pi/n)]\sin(nx)dx.$$

Taking the absolute value, we get the following majorization

$$\left| \int_{\frac{2k\pi}{n}}^{\frac{2k+2}{n}\pi} f(x)\sin(nx)dx \right| \le \int_{\frac{2k\pi}{n}}^{\frac{2k+1}{n}\pi} |f(x) - f(x+\pi/n)|\, dx.$$

Hence

$$\begin{aligned} \left| \int_a^b f(x)\sin(nx)dx \right| &\le \sum_k \int_{\frac{2k\pi}{n}}^{\frac{2k+1}{n}\pi} |f(x) - f(x+\pi/n)|\, dx + |I_a| + |I_b| \\ &\le \int_a^b |f(x) - f(x+\pi/n)|\, dx + |I_a| + |I_b| \end{aligned}$$

Now, let k' be the smallest integer in $A_{a,b}$, then if n increases, also k' changes and the value $\frac{2k'\pi}{n}$ tends to a, hence

$$|I_a| \le \int_a^{\frac{2k'\pi}{n}} |f(x)|\, dx \to 0 \quad \text{when} \quad n \to \infty,$$

and for the same reason also I_b tends to 0 when $n \to \infty$. Applying theorem 10 also the principal term goes to 0 when $n \to \infty$. Hence

$$\int_a^b f(x)\sin(nx)dx \to 0 \quad \text{when} \quad n \to \infty.$$

Lemma 3 (Riemann-Lebesgue, another formulation). *If $f \in L^1(0,2\pi)$, then*

$$a_n(f), b_n(f) \to 0 \quad \textit{when} \quad n \to \infty.$$

Let us give another criterion for the convergence of the Fourier series, which concerns the following definition.

Definition 5. A function f is said to be an *Hölder type function* on $[0,2\pi]$ if

$$|f(x) - f(y)| \le C\,|x-y|^\alpha \quad \forall x,y \in [0,2\pi] \ \text{ with } \ 0 < \alpha \le 1.$$

If f is an Hölder type function on $[0,2\pi]$ we write $f \in C^\alpha([0,2\pi])$.

Exercise 2 *Prove that $C^\alpha([0,2\pi])$ is a Banach space with norm*

$$\|f\|_{C^\alpha} = \sup_x |f(x)| + \sup_{x,y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

Remark 6. Notice that, if $\alpha = 1$, an Hölder type function is a Lipschitz function. Moreover we have the following inclusions

$$C^\alpha([0,2\pi]) \subset C^0([0,2\pi]) \subset L^1([0,2\pi]).$$

Exercise 3 *From the remark it is not implied that $f \in C^{\alpha}([0,2\pi])$ is also differentiable. For istance, prove that the function*

$$x \longmapsto \sqrt{x}$$

is in $C^{\frac{1}{2}}([0,2\pi])$, but it is not differentiable in 0*.*

Another example is

$$f(x) = \sum_{k=1}^{\infty} b^{-k\alpha} \cos(b^k x) \quad \text{with } b \in \mathbb{N} \text{ and } x \in [0,2\pi].$$

This function is in C^{α}, but is not differentiable in any point $x \in [0,2\pi]$.

After this introduction, let us give the following criterion

Theorem 11. *If $f \in C^{\alpha}(\mathbb{R})$ and periodic, then $s_n(f(x)) \to f(x)$ uniformly.*

Proof. Let $f \in C^{\alpha}(\mathbb{R})$ and periodic, then we can compute the Fourier coefficients a_n and b_n and

$$s_n(x) = \int_0^{2\pi} f(y) D_n(x-y) dy.$$

Since $D_n(x)$ is periodic of period 2π and $\int_0^{2\pi} D_n(y)dy = 1$, also $\int_0^{2\pi} D_n(x-y)dy = 1$, hence

$$f(x) = \int_0^{2\pi} f(x) D_n(x-y) dy.$$

Hence

$$\begin{aligned} s_n(x) - f(x) &= \int_0^{2\pi} (f(y) - f(x)) D_n(x-y) dy \\ &= \int_{|x-y|<\delta} (f(y) - f(x)) D_n(x-y) dy + \int_{|x-y|\geq\delta} (f(y) - f(x)) D_n(x-y) dy. \end{aligned}$$

We have only to prove that the two terms go to 0 when $n \to \infty$. For the second term, we have

$$\int_{|x-y|\geq\delta} (f(y) - f(x)) D_n(x-y) dy = \int_{|x-y|\geq\delta} \frac{f(y) - f(x)}{2\pi \sin(\frac{x-y}{2})} \sin\left(\frac{2n+1}{2}(x-y)\right) dy,$$

that converges to 0 when $n \to \infty$ for the Riemann-Lebesgue lemma applied to the function

$$\frac{f(y) - f(x)}{2\pi \sin(\frac{x-y}{2})} \in L^{\infty}(0,2\pi) \subset L^1(0,2\pi).$$

Let us consider the first integral

$$\int_{|x-y|<\delta} \frac{f(y) - f(x)}{\sin(\frac{x-y}{2})} \sin\left(\frac{2n+1}{2}(x-y)\right) dy.$$

Since $f \in C^{\alpha}(\mathbb{R})$,

$$\left| \frac{f(y) - f(x)}{\sin(\frac{x-y}{2})} \right| \leq \frac{2C\,|x-y|^{\alpha}}{|x-y|} \left| \frac{\frac{x-y}{2}}{\sin\frac{x-y}{2}} \right| .$$

The term

$$\left| \frac{\frac{x-y}{2}}{\sin\frac{x-y}{2}} \right|$$

is bounded, while

$$\frac{2C\,|x-y|^{\alpha}}{|x-y|} = \frac{2C}{|x-y|^{1-\alpha}}$$

has a singularity, but this singularity is integrable because $1-\alpha < 1$. Hence $(|x-y|^{1-\alpha})^{-1} \in L^1(0,2\pi)$. So we use the Riemann-Lebesgue lemma to state that also the first integral goes to 0 when $n \to \infty$. Hence $|s_n(x) - f(x)| \to 0$ when $n \to \infty$ for all $x \in [0,2\pi]$. Moreover the convergence is uniform because we consider the difference $|x-y|$that does not dipend on the point x.
Or: Moreover the convergence is uniform, because the convergence in the Riemann-Lebesgue lemma is independent from the bounded interval that we consider, hence it is intependent from the point x.

Chapter 2
Fourier Transform

2.1 Generalities

In the previous chapter we have studied some sufficient conditions so that, if $f \in L^1(0,2\pi)$, we can reconstruct it with its Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)].$$

We want to write this trigonometric series in another way. Let us recall the Euler formulas

$$\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2} \quad \sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i}.$$

Hence

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)] \\ &= \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \frac{\exp(ikx) + \exp(-ikx)}{2} + b_k \frac{\exp(ikx) - \exp(-ikx)}{2i}] \\ &= \frac{a_0}{2} + \frac{1}{2} \sum_{k=1}^{\infty} [(a_k - ib_k) \exp(ikx) + (a_k + ib_k) \exp(-ikx)] \\ &= \frac{1}{2} \sum_{k-\infty}^{+\infty} c_k \exp(ikx), \end{aligned}$$

where $c_k = a_k - ib_k$ if $k > 0$, $c_{-k} = a_{|k|} + ib_{|k|}$ if $k < 0$ and $c_0 = a_0$. Let us consider the case $k \geq 0$. We have

$$\begin{aligned} c_k &= a_k - ib_k = \frac{1}{\pi}\int_0^{2\pi} f(x)[\cos(kx) - i\sin(kx)]dx \\ &= \frac{1}{\pi}\int_0^{2\pi} f(x)\exp(-ikx)dx \end{aligned}$$

If $k < 0$ we have

$$\begin{aligned} c_k &= a_{|k|} + ib_{|k|} = \frac{1}{\pi}\int_0^{2\pi} f(x)[\cos(|k|x) + i\sin(|k|x)]dx \\ &= \frac{1}{\pi}\int_0^{2\pi} f(x)\exp(i|k|x)dx \\ &= \frac{1}{\pi}\int_0^{2\pi} f(x)\exp(-ikx)dx \end{aligned}$$

and the same holds also for $k = 0$. Hence

$$c_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\exp(-ikx)dx \quad \text{for all } k \in \mathbb{Z}.$$

If f is periodic we can substitute the interval $[0, 2\pi]$ with the interval $[-\pi, \pi]$ and define the number

$$\hat{f}_k = \int_{-\pi}^{\pi} f(x)\exp(-ikx)dx \quad \Rightarrow \quad f(x) = \frac{1}{2\pi}\sum_{k-\infty}^{+\infty} \hat{f}_k \exp(ikx). \tag{2.1}$$

This result on a periodic function, suggests us a way to extend this notation to a non periodic function

Definition 6. Let $f \in L^1(\mathbb{R})$. The *Fourier transform* of f is

$$\hat{f}(\xi) := \int_{\mathbb{R}} f(x)\exp(-i\xi x)dx.$$

We will return later on the Fourier transform. Now we want to give a second way to find an analogous transformation.

Let $f \in L^1(-T/2, T/2)$ be a periodic function with period $T > 0$. We know a formula (2.1) to represent a function in $L^1(-\pi, \pi)$. Applying to f the following substitution: for $y \in [-T/2, T/2]$ take $y = \frac{T}{2\pi}x$ with $x \in [-\pi, \pi]$, we obtain a function $g(x) = f(\frac{T}{2\pi}x)$, that is in $L^1(-\pi, \pi)$. Hence we can use the previous theory and, using $x = \frac{2\pi}{T}y$, state that

$$\hat{g}_k = \int_{-\pi}^{\pi} g(x)\exp(-ikx)dx = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(y)\exp(-ik\frac{2\pi}{T}y)\frac{2\pi}{T}dy \;\Rightarrow\; g(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{g}_k \exp(ikx).$$

Then

Fig. 2.1 Jean Baptiste Joseph Fourier (1768-1830)

$$f(y) = g(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \hat{g}_k \exp(ik\frac{2\pi}{T}y).$$

If we define

$$\hat{f}_k := \int_{-T/2}^{T/2} f(y)\exp(-ik\frac{2\pi}{T}y)dy,$$

we get

$$\hat{g}_k = \frac{2\pi}{T}\hat{f}_k \quad \Rightarrow \quad f(y) = \frac{1}{T}\sum_{k\in\mathbb{Z}} \hat{f}_k \exp(ik\frac{2\pi}{T}y).$$

So we have obteined the Fourier series with period $T \neq 2\pi$.

Let $f \in L^1(\mathbb{R})$ be a non periodic function. We can consider f in the interval $[-T/2, T/2]$ and extend it periodically on the whole $\mathbb{R}$. Then we can apply to this function the previous theory. Since sending T to infinity, we obtain the original function f, we can extend the theory to a general non periodic $f \in L^1(\mathbb{R})$.
Fix ξ and choose $k_T \in \mathbb{Z}$ such that $\frac{k_T}{T} \leq \xi < \frac{k_T+1}{T}$. If $T \to \infty$ also $k_T \to \infty$ and $\frac{k_T}{T} \to \xi$.
Then taking the limit for $T \to \infty$, we obtain

$$\hat{f}_k = \int_{-T/2}^{T/2} f(y)\exp(-ik\frac{2\pi}{T}y)dy \longrightarrow \int_{\mathbb{R}} f(y)\exp(-2\pi i\xi y)dy =: \hat{f}(\xi)$$

$$f(y) = \frac{1}{T}\sum_{k\in\mathbb{Z}} \hat{f}_k \exp(ik\frac{2\pi}{T}y) \longrightarrow \int_{\mathbb{R}} \hat{f}(\xi)\exp(2\pi i\xi y)d\xi =: f(y)$$

that is another way to introduce the Fourier transform.

Return now to the first definition of Fourier transform: let $f \in L^1(\mathbb{R})$,

$$\hat{f}(\xi) = \int_{\mathbb{R}} \exp(-ix\xi) f(x) dx.$$

Let us observe that, if $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R}) = \{\varphi : \mathbb{R} \to \mathbb{C} \text{ continuous: } \lim_{x\to\infty} f(x) = 0\}$. Hence we can consider the Fourier transform as an operator

$$\begin{array}{rcl} \mathscr{F} : L^1(\mathbb{R}) & \longrightarrow & C_0(\mathbb{R}) \\ f & \longmapsto & (\mathscr{F} f)(\xi) = \hat{f}(\xi) \end{array}$$

It is easy to see that $\mathscr{F}$ is a linear operator, i.e. $\mathscr{F}(\lambda f + g) = \lambda \mathscr{F}(f) + \mathscr{F}(g)$.

Exercise 4 *Prove that:*

(i) $C_0(\mathbb{R})$ *with the norm* $\|\cdot\|_\infty$ *is a complex Banach space;*
(ii) Every $f \in C_0(\mathbb{R})$ *is uniformly continuous;*
(iii) $C_c^\infty(\mathbb{R})$ *is dense in* $C_0(\mathbb{R})$*, where* $C_c^\infty(\mathbb{R})$ *is the set of the* C^∞ *function with compact support;*
(iv) If $f, g \in C_0(\mathbb{R})$*, then* $fg \in C_0(\mathbb{R})$*;*
(v) $C_0(\mathbb{R})$ *is a commutative Banach algebra.*

Exercise 5 *Prove that:*

(i) $L^1(\mathbb{R})$ *with the norm* $\|\cdot\|_1$ *is a complex Banach space;*
(ii) $C_c^\infty(\mathbb{R})$ *is dense in* $L^1(\mathbb{R})$ *with respect to the norm* $\|\cdot\|_1$*;*
(iii) If $f, g \in L^1(\mathbb{R})$*,* $f * g \in L^1(\mathbb{R})$ *where* $*$ *is the convolution product;*
(iv) $L^1(\mathbb{R})$ *is a Banach algebra with respect to the convolution product.*

Since if $f, g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$,we can compute

$$\widehat{f * g}(\xi) = \int_{\mathbb{R}} \exp(-ix\xi)(f * g)(x) dx = \int_{\mathbb{R}} \exp(-ix\xi) \left[\int_{\mathbb{R}} f(x-y) g(y) dy \right] dx$$

We can apply the Fubini-Tonelli's theorem, because

$$|\exp(-ix\xi) f(x-y) g(y)| = |f(x-y) g(y)| \quad \text{and} \quad \exp(-ix\xi) f(x-y) g(y) \in L^1(\mathbb{R} \times \mathbb{R}),$$

hence

$$\begin{aligned}
\widehat{f*g}(\xi) &= \int_{\mathbb{R}} \exp(-ix\xi) \left[\int_{\mathbb{R}} f(x-y)g(y)dy\right] dx \\
&= \int_{\mathbb{R}} g(y) \left[\int_{\mathbb{R}} \exp(-ix\xi) f(x-y)dx\right] dy \\
&= \int_{\mathbb{R}} \exp(-iy\xi)g(y) \left[\int_{\mathbb{R}} \exp(-i(x-y)\xi) f(x-y)dx\right] dy \\
&= \left(\int_{\mathbb{R}} \exp(-iy\xi)g(y)dy\right) \left(\int_{\mathbb{R}} \exp(-iz\xi) f(z)dz\right) \\
&= \hat{f}(\xi)\hat{g}(\xi)
\end{aligned}$$

where we have used the substitution $x-y=z$. Hence we have proved the following theorem

Theorem 12. $\mathscr{F} : L^1(\mathbb{R})_{(+,\lambda,*)} \to C_0(\mathbb{R})_{(+,\lambda,\cdot)}$ *is an endomorphism, that is* $\widehat{f*g} = \hat{f}\cdot\hat{g}$.

2.2 The Schwartz Space

Once we have the Fourier transform of a function in L^1, we would like to reconstruct it. To do that we have to introduce the *inverse Fourier transform*. A candidate is given by

$$\frac{1}{2\pi}\int_{\mathbb{R}} g(\xi)\exp(ix\xi)d\xi.$$

Let us note that this function is well defined for all $g \in L^1(\mathbb{R})$. So we can define $\widetilde{\mathscr{F}} : L^1 \to C_0$ such that

$$(\widetilde{\mathscr{F}}g)(x) = \check{g}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} g(\xi)\exp(ix\xi)d\xi.$$

We require that $\widetilde{\mathscr{F}}\mathscr{F} = Id$. We observe immediatly however that this is not possible, because $\mathscr{F} : L^1 \to C_0$, and $\widetilde{\mathscr{F}}$ isn't defined on the whole space C_0. An example of that is the following

Example 4. Let

$$f(x) = \begin{cases} 1 \text{ if } x \in [-a,a] \\ 0 \text{ otherwise} \end{cases}$$

Since $f \in L^1$ we can compute the Fourier transform

$$\begin{aligned}(\mathscr{F}f)(\xi) &= \int_{\mathbb{R}} \exp(-ix\xi)f(x)dx = \int_{-a}^{a} \exp(-ix\xi)dx \\ &= \int_{-a}^{a} (\cos(x\xi) - i\sin(x\xi))dx = \int_{-a}^{a} \cos(x\xi)dx \\ &= \frac{2}{\xi}\sin(a\xi)\end{aligned}$$

but $\hat{f}(\xi) \notin L^1$ and we can't apply $\widetilde{\mathscr{F}}$ to it.

We could try to restrict $\mathscr{F}$ to the space $L^1 \cap C_0$, but $\mathscr{F}(L^1 \cap C_0) \nsubseteq L^1 \cap C_0$.
The only way is to introduce a new subspace: the *Schwartz Space*

$$\mathscr{S} = \left\{ f \in C^\infty(\mathbb{R}),\ \forall\, n,m \geq 0\ \left| x^m f^{(n)}(x) \right| \leq C_{n,m} \right\}$$

This space is such that $\mathscr{F}(\mathscr{S}) \subseteq \mathscr{S}$.

Remark 7. Notice that $\mathscr{S} \subset L^1(\mathbb{R}) \cap C_0(\mathbb{R})$, indeed taking $n = 0$ and $m = 0$ we obtain $|f(x)| \leq C_{0,0}$ that is, f is bounded and with $m = 2$ we get $|x^2 f(x)| \leq C_{0,2}$. Then

$$|f(x)| \leq \frac{C}{1+x^2} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}).$$

It is trivial that $C_c^\infty(\mathbb{R}) \subset \mathscr{S}$, but this set is too small for our goal, because $\mathscr{F}(C_c^\infty(\mathbb{R})) \nsubseteq C_c^\infty(\mathbb{R})$.
However, since $C_c^\infty(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, we have that $\mathscr{S}$ is dense in $L^1(\mathbb{R})$ and the following inclusions hold

$$C_c^\infty(\mathbb{R}) \subset \mathscr{S} \subset C_0(\mathbb{R}) \cap L^1(\mathbb{R})$$

Exercise 6 *Prove that*

(i) *$\mathscr{S}$ is a vector space over $\mathbb{C}$;*
(ii) *if $f, g \in \mathscr{S}$, $fg \in \mathscr{S}$, that is $\mathscr{S}$ is an algebra;*
(iii) *if $f, g \in \mathscr{S}$, $f * g \in \mathscr{S}$;*
(iv) *if $f \in \mathscr{S}$, $\mathscr{F}f \in \mathscr{S}$.*

Hence we can define

$$\mathscr{F} : \mathscr{S} \longrightarrow \mathscr{S} \quad \text{and} \quad \widetilde{\mathscr{F}} : \mathscr{S} \longrightarrow \mathscr{S}$$

Example 5. Let $f(x) = \exp(-ax^2)$, $a > 0$. $f \in \mathscr{S}$ hence we can compute the Fourier transform and then the inverse Fourier transform. We have that

$$(\mathscr{F}f)(\xi) = \int_{\mathbb{R}} \exp(-ix\xi - ax^2)dx = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{\xi^2}{4a}\right)$$

$$(\widetilde{\mathscr{F}}\mathscr{F}f)(x) = \exp(-ax^2) = f(x)$$

Hence in this case we have that $\widetilde{\mathscr{F}}\mathscr{F} = Id$. Our goal will be to prove this in general.

Example 6. Let $f \in L^1(\mathbb{R})$ such that $f' \in L^1(\mathbb{R})$. We can compute the Fourier transform of f', that is

$$\begin{aligned}(\mathscr{F} f')(\xi) &= \int_{\mathbb{R}} \exp(-ix\xi) f'(x)dx \\ &= [f(x)\exp(-ix\xi)]_{-\infty}^{+\infty} + i\xi \int_{\mathbb{R}} \exp(-ix\xi) f(x)dx \\ &= i\xi(\mathscr{F} f)\end{aligned}$$

where we have used that $f(+\infty), f(-\infty) < +\infty$, because $f' \in L^1$, and then $f(+\infty) = f(-\infty) = 0$ because $f \in L^1$. Hence we have obtained the following main formula for the Fourier transform of a derivation:

$$(\mathscr{F} f')(\xi) = i\xi(\mathscr{F} f)$$

We want now to prove that $\widetilde{\mathscr{F}}\mathscr{F} = Id$ on $\mathscr{S}$. To reach our goal we start with the following proposition

Proposition 3. *The following statements hold:*

(i) $2\pi\mathscr{F}(f\check{g}) = \hat{f} * g$ *for all* $f, g \in L^1(\mathbb{R})$*;*
(ii) $\widetilde{\mathscr{F}}(f\hat{g}) = \check{f} * g$ *for all* $f, g \in L^1(\mathbb{R})$.

Proof. We will give the proof only of statement (i). The proof of statement (ii) is analogous.
Let $f, g \in L^1(\mathbb{R})$, then $\hat{f}, \hat{g}, \check{f}, \check{g} \in C_0(\mathbb{R})$ and are bounded. Hence $f\check{g} \in L^1(\mathbb{R})$, because $|f\check{g}| \leq c|f| \in L^1(\mathbb{R})$, and we can apply the Fourier transform

$$\mathscr{F}(f\check{g})(\xi) = \int_{\mathbb{R}} \exp(-ix\xi) f(x)\check{g}(x)dx = \int_{\mathbb{R}} \exp(-ix\xi) f(x) \left[\frac{1}{2\pi}\int_{\mathbb{R}} \exp(ixy)g(y)dy\right] dx$$

Now $|\exp(-ix\xi)\exp(ixy)f(x)g(y)| = |f(x)g(y)|$ and $\exp(-ix\xi)\exp(ixy)f(x)g(y) \in L^1(\mathbb{R}^2)$, hence we can apply the Fubini-Tonelli's theorem obtaining

$$\begin{aligned}\mathscr{F}(f\check{g})(\xi) &= \frac{1}{2\pi}\int_{\mathbb{R}} g(y)\left[\int_{\mathbb{R}} \exp(-ix(\xi - y))f(x)dx\right] dy \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} g(y)\hat{f}(\xi - y)dy = \frac{1}{2\pi}(g * \hat{f})(\xi).\end{aligned}$$

Theorem 13. $\mathscr{F}\widetilde{\mathscr{F}} = Id$ *on* $\mathscr{S}$.

Proof. Applying statement (i) of proposition 3 with $f(x) = \exp(-ax^2)$, we have $\mathscr{F}(f\check{g}) = \frac{1}{2\pi}\hat{f} * g$, where

$$\mathscr{F}(f\check{g}) = \int_{\mathbb{R}} \exp(-ix\xi - ax^2)(\widetilde{\mathscr{F}} g)(x)dx \tag{2.2}$$

and

$$\frac{1}{2\pi}\hat{f}*g=\frac{1}{2\pi}\int_{\mathbb{R}}\sqrt{\frac{\pi}{a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)g(y)dy=\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)g(y)dy. \tag{2.3}$$

When $a\to 0$, we have that the first integral is

$$\int_{\mathbb{R}}\exp(-ix\xi)(\widetilde{\mathscr{F}}g)(x)dx=\mathscr{F}\widetilde{\mathscr{F}}g(\xi). \tag{2.4}$$

If we prove that

$$\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)g(y)dy\to g(\xi),$$

we will obtain that $\mathscr{F}\widetilde{\mathscr{F}}g(\xi)=g(\xi)$. Let us prove the last statement.
Consider

$$\begin{aligned}
|\Delta_a| &= \left|\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)g(y)dy-g(\xi)\right| \\
&= \left|\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)(g(y)-g(\xi))dy\right| \\
&\le \int_{\mathbb{R}}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)|g(y)-g(\xi)|\,dy \\
&= \int_{|\xi-y|<\delta}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)|g(y)-g(\xi)|\,dy+ \\
&\quad +\int_{|\xi-y|\ge\delta}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)|g(y)-g(\xi)|\,dy.
\end{aligned}$$

Since $g\in\mathscr{S}$, g is bounded, that is $\exists M\in\mathbb{R}$ such that $|g(y)|\le M$ for each y, and g is continuous, then for each $\varepsilon>0$ there exists $\delta>0$ such that $|g(y)-g(\xi)|<\varepsilon,\ \forall|\xi-y|<\delta$. Hence

$$\begin{aligned}
|\Delta_a| &\le \varepsilon\int_{|\xi-y|<\delta}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)dy+2M\int_{|\xi-y|\ge\delta}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{(\xi-y)^2}{4a}\right)dy \\
&\le \varepsilon\cdot 1+\int_{|y|\ge\delta}\frac{1}{\sqrt{4\pi a}}\exp\left(-\frac{y^2}{4a}\right)dy
\end{aligned}$$

The second term tends to 0 as a goes to 0, because it is the integral on $(-\infty,-\delta]\cup[\delta,\infty)$ of a gaussian with mean $a<<1$. Hence $\limsup_{a\to 0}|\Delta_a|\le\varepsilon\ \ \forall\varepsilon>0$ that proves that there exists

$$\lim_{a\to 0}|\Delta_a|=0.$$

Then

$$\lim_{a\to 0}\int_{\mathbb{R}} \frac{1}{\sqrt{4\pi a}} \exp\left(-\frac{(\xi-y)^2}{4a}\right) g(y)dy = g(\xi).$$

This result together with (2.2), (2.3), (2.4) and proposition 3 prove that

$$\mathscr{F}\circ\widetilde{\mathscr{F}}(g) = Id(g) \quad \forall g\in\mathscr{S}.$$

Remark 8. Someone could ask, why haven't we done the conputation directly in this proof? More precisely, why haven't we calculated

$$\mathscr{F}\widetilde{\mathscr{F}}g(\xi) = \int_{\mathbb{R}} \exp(-ix\xi)\left(\frac{1}{2\pi}\int_{\mathbb{R}} \exp(ixy)g(y)dy\right)dx$$

The problem is that

$$|\exp(-ix\xi)\exp(ixy)g(y)| = |g(y)|$$

that isn't in $L^1(\mathbb{R}^2)$, then we can't change the order of the integrals.

Exercise 7 *Prove theorem 13, using the equation* $\widetilde{\mathscr{F}}(f\hat{g}) = \check{f} * g$.

On the space $\mathscr{S}$ we can define an inner product (or scalar product) as follows: if $f,g\in\mathscr{S}$ then

$$\langle f,g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}dx$$

and a norm

$$\|f\|_2^2 := \langle f,f\rangle = \int |f(x)|^2 dx$$

Exercise 8 *From the definition of the inner product follows* $\langle \hat{f},\hat{g}\rangle = \int_{\mathbb{R}} \hat{f}(x)\overline{\hat{g}(x)}dx$, *prove that*

$$\int_{\mathbb{R}} \hat{f}(x)\overline{\hat{g}(x)}dx = 2\pi\int_{\mathbb{R}} f(x)\overline{g(x)}dx$$

that is the Fourier transform preserves the scalar product:

$$\langle \hat{f},\hat{g}\rangle = 2\pi\langle f,g\rangle$$

and also the norm

$$\left\|\hat{f}\right\|_2^2 = 2\pi\|f\|_2^2.$$

This result allows us to extend the Fourier transform to the functions in L^2. Indeed, since $\mathscr{S}\subset L^2(\mathbb{R})$ is dense, for each $f\in L^2(\mathbb{R})$, we can find a sequence $(\varphi_n)_n\subset\mathscr{S}$ that converges to f in the L^2-sense. Then $(\varphi_n)_n$ is a Cauchy sequence and

$$\|\varphi_n-\varphi_m\|_{L^2} = \frac{1}{2\pi}\|\hat{\varphi}_n-\hat{\varphi}_m\|_{L^2} \longrightarrow 0.$$

This means that also $(\hat{\varphi}_n)_n\subset\mathscr{S}$ is a Cauchy sequence in $L^2(\mathbb{R})$ and converges to a $g\in L^2(\mathbb{R})$. Then we can define $\mathscr{F}(f) := g$. We leave as an exercise that this definition is independent of the choice of the sequence $(\varphi_n)_n$.
Hence we have defined

$$\mathscr{F}: L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \quad \text{and} \quad \widetilde{\mathscr{F}}: L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

Example 7 (Exercise). Let f be as in example (4), then $f \in L^2 \subset L^1$, $\hat{f} = \frac{2\sin(a\xi)}{\xi} \in L^2$ and

$$\|f\|_2^2 = \frac{1}{2\pi} \left\| \frac{2\sin(a\xi)}{\xi} \right\|_2^2.$$

In this way we can compute

$$\int_{\mathbb{R}} \frac{\sin(x)}{x}\, dx = \int_{\mathbb{R}} \frac{\sin^2(x)}{x^2}\, dx = \pi.$$

2.3 The dual space

Let V be a vector space. The *algebraic dual space* of V on $\mathbb{C}$ (or a field $\mathbb{K}$) is

$$V^+ := \{T : V \to \mathbb{C} : \text{ linear}\}$$

V^+ is a vector space on $\mathbb{C}$, with the operations

$$(\lambda T + S)(v) = \lambda T(v) + S(v) \quad \forall T, S \in V^+, \forall v \in V.$$

We can define also the algebraic dual space of V^+, that is

$$V^{++} : \{S : V^+ \to \mathbb{C} : \text{ linear}\}.$$

It holds $V \subseteq V^{++}$, indeed for all $v \in V$, define $v(T) := T(v) \quad \forall T \in V^+$. Then v defines a linear functional in V^{++}.
There is also another kind of dual space, that is defined on topological vector spaces.

Definition 7. A *topological vector space* V on $\mathbb{C}$ is a vector space on $\mathbb{C}$, that is also a topological space, such that the operations

$$V \times V \longrightarrow V : (v_1, v_2) \longmapsto v_1 + v_2$$
$$\mathbb{C} \times V \longrightarrow V : (\lambda, v) \longmapsto \lambda v$$

are continuous.

Exercise 9 *To be a topological vector space is less restrictive then to be a Banach space. Indeed prove that*

- *a Banach space is a topological vector space;*
- $\mathscr{S}$ *is not a Banach space, but it is a topological vector space.*

Definition 8. The *dual space* of a topological vector space is defined as

$$V^{'} := \{T : V \to \mathbb{C} : \text{ linear and continuous}\} \subset V^+$$

It's easy to see that V' is still a topological vector space, then we can take its dual V''. V is always contained in V''. If it holds also $V = V''$, V is called *reflexive*.
On $\mathscr{S}$ we introduce the following functions:

$$\forall n,m \in \mathbb{N} \qquad \begin{aligned} P_{n,m} : \mathscr{S} &\longrightarrow \mathbb{R}^+ \\ \varphi &\longmapsto \sup_{x\in\mathbb{R}} |x^m \mathrm{D}^n \varphi| \ (\leq C_{m,n}) \end{aligned}$$

Proposition 4. *For each $n,m \in \mathbb{N}$, $P_{n,m}$ is a seminorm, that is*

(i) $P_{n,m}(\varphi+\psi) \leq P_{n,m}(\varphi) + P_{n,m}(\psi), \quad \forall \varphi, \psi \in \mathscr{S}$

(ii) $P_{n,m}(\lambda\varphi) = |\lambda| P_{n,m}(\varphi), \quad \forall \lambda \in \mathbb{R}, \varphi \in \mathscr{S}$

Moreover, the following function

$$d(\varphi,\psi) := \sum_{n,m} \frac{P_{n,m}(\varphi-\psi)}{1+P_{n,m}(\varphi-\psi)} \cdot \frac{1}{2^{n+m}}$$

defines a distance on $\mathscr{S}$.

Proof. The first statement follows from

$$|x^m \mathrm{D}^n(\varphi+\psi)| = |x^m \mathrm{D}^n \varphi + x^m \mathrm{D}^n \psi| \leq |x^m \mathrm{D}^n \varphi| + |x^m \mathrm{D}^n \psi|.$$

The second statement follows immediatly from the definition.
It remains to prove that $d(\varphi,\psi)$ is a distance, that is

(i) $d(\varphi,\psi) \geq 0 \quad \forall \varphi,\psi \in \mathscr{S}$

(ii) $d(\varphi,\psi) \leq d(\varphi,\rho) + d(\rho,\psi) \quad \forall \varphi,\psi,\rho \in \mathscr{S}$

(iii) $d(\varphi,\psi) = d(\psi,\varphi) \quad \forall \varphi,\psi \in \mathscr{S}$

The first and the third properties are clear. To prove the second one, define

$$x := P_{n,m}(\varphi-\psi) \geq 0 \quad \text{and} \quad y := P_{n,m}(\varphi-\rho) + P_{n,m}(\rho-\psi) \geq 0.$$

Since $P_{n,m}$ is a seminorm, $y - x \geq 0$ and $\frac{y}{1+y} - \frac{x}{1+x} = \frac{y+xy-xy-x}{(1+y)(1+x)} \geq 0$, that is

$$\frac{x}{1+x} \leq \frac{y}{1+y}.$$

Hence

$$
\begin{aligned}
d(\varphi,\psi) &\leq \sum_{n,m} \frac{P_{n,m}(\varphi-\rho)+P_{n,m}(\rho-\psi)}{1+P_{n,m}(\varphi-\rho)+P_{n,m}(\rho-\psi)} \cdot \frac{1}{2^{n+m}} \\
&= \sum_{n,m} \frac{P_{n,m}(\varphi-\rho)}{1+P_{n,m}(\varphi-\rho)+P_{n,m}(\rho-\psi)} \cdot \frac{1}{2^{n+m}} + \\
&\quad + \sum_{n,m} \frac{P_{n,m}(\rho-\psi)}{1+P_{n,m}(\varphi-\rho)+P_{n,m}(\rho-\psi)} \cdot \frac{1}{2^{n+m}} \\
&\leq \sum_{n,m} \frac{P_{n,m}(\varphi-\rho)}{1+P_{n,m}(\varphi-\rho)} \cdot \frac{1}{2^{n+m}} + \sum_{n,m} \frac{P_{n,m}(\rho-\psi)}{1+P_{n,m}(\rho-\psi)} \cdot \frac{1}{2^{n+m}} \\
&= d(\varphi,\rho)+d(\rho,\psi)
\end{aligned}
$$

Exercise 10 *Prove that $\mathscr{S}$ is a topological vector space, with respect to the topology induced by the distance, and is complete, that means that for each sequence $(x_n)_n$ such that $d(x_n,x_m) \to 0$, there exists $x \in \mathscr{S}$ such that $d(x_n,x) \to 0$.*

Since $\mathscr{S}$ is a topological vector space, its dual space $\mathscr{S}'$ is well defined. From the Fourier transform given on $\mathscr{S}$, we can define $\mathscr{F} : \mathscr{S}' \longrightarrow \mathscr{S}'$ as follows: for each $T \in \mathscr{S}'$

$$(\mathscr{F}T)(\varphi) := T(\mathscr{F}\varphi) \quad \forall \varphi \in \mathscr{S}$$

For each function $f \in L^2$ we can associate the functional $T_f(\varphi) := \int_{\mathbb{R}} f(x)\varphi(x)dx$ in $\mathscr{S}'$ because if $g \in \mathscr{S}$, $f \cdot g \in L^2$. Hence $L^2 \subset \mathscr{S}'$. Since $\mathscr{S} \subset L^2$, it holds also $\mathscr{S} \subset \mathscr{S}'$, where to each $\psi \in \mathscr{S}$ we associate

$$T_\psi(\varphi) := \int_{\mathbb{R}} \psi(x)\varphi(x)dx.$$

Hence we desire that

Proposition 5. *For each $\psi \in \mathscr{S}$*

$$\mathscr{F}\psi = \mathscr{F}T_\psi$$

Proof. Let us consider the definition of $\mathscr{F}$

$$(\mathscr{F}T_\psi)(\varphi) = T_\psi(\mathscr{F}\varphi) = \int_{\mathbb{R}} \psi(\xi)(\mathscr{F}\varphi)(\xi)d\xi = \int_{\mathbb{R}} \psi(\xi)\hat{\varphi}(\xi)d\xi$$

Now

$$\int_{\mathbb{R}} \psi(\xi)\hat{\varphi}(\xi)d\xi = \int_{\mathbb{R}} \psi(\xi)\left(\int_{\mathbb{R}} \exp(-ix\xi)\varphi(x)dx\right)d\xi = \int_{\mathbb{R}} \hat{\psi}(x)\varphi(x)dx$$

where in the second equality we have applied Fubini Tonelli's theorem. Hence

$$(\mathscr{F}T_\psi)(\varphi) = \int_{\mathbb{R}} \hat{\psi}(x)\varphi(x)dx = T_{\hat{\psi}}(\varphi).$$

On the other hand

$$\mathscr{F}\psi = \int_{\mathbb{R}} \exp(-ix\xi)\psi(x)dx = \hat{\psi} = T_{\hat{\psi}}$$

The latter two results together complete the proof.

Example 8. Let $f \equiv 1$ and consider for each $\varphi \in \mathscr{S}$

$$T_1(\varphi) = \int_{\mathbb{R}} \varphi(x)dx$$

For the last proposition

$$\hat{T}_1(\varphi) = T_1(\hat{\varphi}) = \int_{\mathbb{R}} \hat{\varphi}(\xi)d\xi$$

and, since $2\pi\varphi(x) = \int \exp(ix\xi)\hat{\varphi}(\xi)d\xi$, we obtain

$$\hat{T}_1(\varphi) = 2\pi\varphi(0)$$

On the other hand, if we consider the Dirac function δ_0,

$$T_{\delta_0}(\psi) := \int \varphi(x)\delta_0(dx) = \varphi(0)$$

Hence

$$\hat{T}_1 = 2\pi\delta_0$$

Example 9. Let $f \in L^2(\mathbb{R})$, we want to prove that

$$\hat{T}_f = T_{\hat{f}}.$$

We start by proving this statement in the case $f \in \mathscr{S}$. For each $\varphi \in \mathscr{S}$,

$$\hat{T}_f(\varphi) = T_f(\hat{\varphi}) = \int_{\mathbb{R}} f(\xi)\hat{\varphi}(\xi)d\xi = \int_{\mathbb{R}} \hat{f}(\xi)\varphi(\xi)d\xi = T_{\hat{f}}(\varphi)$$

In the general case, if $\varphi, \psi \in L^2$, taking two sequences $(\varphi_n)_n, (\psi_n)_n \subset \mathscr{S}$ converging respectively to φ and ψ and applying the dominated convergence theorem, we can prove that

$$\int_{\mathbb{R}} \varphi(x)\hat{\psi}(x)dx = \int_{\mathbb{R}} \hat{\varphi}(x)\psi(x)dx \quad \forall\varphi, \psi \in L^2(\mathbb{R})$$

that completes the proof. Notice that all the integrals are well-defined, because the functions are in L^2.

We want to give now a definition of derivative for a functional $T \in \mathscr{S}'$. Let us call $C_c^\infty(\mathbb{R}) = \mathscr{D}(\mathbb{R})$. Then $\mathscr{D} \subset \mathscr{S} \subset \mathscr{S}' \subset \mathscr{D}'$ and $\mathscr{D}'$ is called the space of the *distribution in the sense of L. Schwartz*. In the theory of the distribution is defined a derivation: for all $T \in \mathscr{D}'$,

$$(\mathrm{D}T)(\varphi) = -T(\varphi')$$

Then also for $T \in \mathscr{S}' \subset \mathscr{D}'$ we define

$$\mathrm{D}T(\varphi) := -T(\varphi^{'}) \quad \forall \varphi \in \mathscr{S}$$

and since $\hat{T} \in \mathscr{S}^{'}$ we have

$$\mathrm{D}\hat{T}(\varphi) = -\hat{T}(\varphi^{'}) = -T(\mathscr{F}(\varphi^{'}))$$

Example 10. With this definition we can calculate $(\mathrm{D}\delta_0)(\varphi) = -\delta_0(\mathrm{D}\varphi) = -\varphi^{'}(0)$.

The reason to put a $-$ in this definition is the following: if we take $f \in \mathscr{S}$

$$T_f^{'}(\varphi) = -T_f(\varphi^{'}) = -\int_{\mathbb{R}} f(\xi)\varphi^{'}(\xi)d\xi = [f(\xi)\varphi(\xi)]_{-\infty}^{+\infty} + \int_{\mathbb{R}} f^{'}(\xi)\varphi(\xi)d\xi$$

But $[f(\xi)\varphi(\xi)]_{-\infty}^{+\infty} = 0$, because $f, \varphi \in \mathscr{S}$, then

$$T_f^{'}(\varphi) = \int_{\mathbb{R}} f^{'}(\xi)\varphi(\xi)d\xi = T_{f^{'}}(\varphi).$$

Then for each $f \in \mathscr{S}$ we have the following property

$$T_f^{'} = T_{f^{'}}.$$

Example 11. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ \alpha & \text{if } x = 0, \quad \text{where } \alpha \in \mathbb{R} \\ 0 & \text{if } x < 0 \end{cases}$$

Since f is not derivable in 0, we can't compute the derivative with the last property. However we can compute it with the thoery of the distributions: for each $\varphi \in \mathscr{S}$

$$T_f(\varphi) = \int_{\mathbb{R}} f(\xi)\varphi(\xi)d\xi = \int_{-\infty}^{0} 0 \cdot \varphi(\xi)d\xi + \int_{0}^{+\infty} 1 \cdot \varphi(\xi)d\xi = \int_{0}^{+\infty} \varphi(\xi)d\xi$$

and the derivative in the sense of the distributions is

$$T_f^{'}(\varphi) = -\int_{\mathbb{R}} f(\xi)\varphi(\xi)d\xi = -\int_{0}^{+\infty} \varphi(\xi)d\xi = \varphi(0)$$

Hence we have calculated that $T_f^{'} = \delta_0$.

Example 12. We make a generalization of the previous example. Let

$$g(x) = \begin{cases} 1 & \text{if } x > a \\ \alpha & \text{if } -a \leq x \leq a, \quad \text{where } \alpha \in \mathbb{R} \\ 0 & \text{if } x < -a \end{cases}$$

We want to prove that $T_g^{'} = \delta_{-a} - \delta_a$.
Indeed, considering the function f of the previous example,

$$g(x) = f(x+a) - f(x-a) := f_a(x) - f_{-a}(x).$$

Then $T_g^{'} = T_{f_a - f_{-a}}^{'}$.
Now, in general for each $h_1, h_2, \varphi \in \mathscr{S}$, $T_{h_1+h_2}(\varphi) = \int (h_1 + h_2)\varphi d\xi = T_{h_1}(\varphi) + T_{h_2}(\varphi)$. Hence

$$T_g^{'} = (T_{f_a} - T_{f_{-a}})^{'} = T_{f_a}^{'} - T_{f_{-a}}^{'} = \delta_{-a} - \delta_a$$

Exercise 11 *Let $a \in \mathbb{R}$ and consider*

$$f(x) = \begin{cases} \frac{x}{a} + 1 & \text{if } -a < x < 0 \\ -\frac{x}{a} + 1 & \text{if } 0 \le x < a \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{1}{a} & \text{if } -a < x < 0 \\ -\frac{1}{a} & \text{if } 0 \le x < a \\ 0 & \text{otherwise} \end{cases}$$

Prove that $T_f^{'} = T_g$.
Notice that $f \in L^p$ and $f' \in L^p$ too. (This is an analoguos to the Sobolev spaces)

2.4 Heisenberg Principle

Let $f \in L^2(\mathbb{R};\mathbb{C})$ and $\hat{f} = \mathscr{F}(f) \in L^2$ ([1]). Recall that $\mathscr{F}$ preserves the inner product

$$\int_{\mathbb{R}} |f|^2 dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}|^2 d\xi$$

Definition 9. If $x\frac{|f(x)|^2}{\|f\|_2^2}$ is integrable, we can define the *mean* of x

$$\langle x \rangle := \int_{\mathbb{R}} x \frac{|f(x)|^2}{\|f\|_2^2} dx$$

and, using the Fourier transform, the mean of ξ:

$$\langle \xi \rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \xi \frac{|\hat{f}(\xi)|^2}{\|f\|_2^2} d\xi$$

If the integrals exist, we can also calculate the variance:

$$|\Delta x|^2 := \int (x - \langle x \rangle)^2 \frac{|f(x)|^2}{\|f\|_2^2} dx \quad \text{and} \quad |\Delta \xi|^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (\xi - \langle \xi \rangle)^2 \frac{|\hat{f}(\xi)|^2}{\|f\|_2^2} d\xi$$

We want to prove the following fundamental principle:

[1] Untill now we have done all the theory on $\mathbb{R}$, but it can be generalized without any problem to $\mathbb{R}^n$. For example

$$\exp(ix\xi) \quad \text{becomes} \quad \exp(i\langle x, \xi \rangle)$$

Theorem 14 (Heisenberg Principle). *Let $f \in L^2(\mathbb{R};\mathbb{C})$. Then*

$$\Delta x \Delta \xi \geq \frac{1}{2\pi}$$

This is the uncertainty principle; indeed, if we consider a physical example, as an electron, it says that we can't have in the same time a precise measure of the position, x, *and* of the impulse of the electron.

Example 13. Consider an electron whose distribution (moltiplied for a suitable constant) is the gaussian function $f(x) = \exp(-ax^2)$. From example 5 we know that $\hat{f}(\xi) = \frac{\pi}{a}\exp(-\frac{\xi^2}{4a})$. Then if a is very small, we can have a accurate measure of the position of the electron, but the Fourier transform, that represents the impulse of the electron, doesn't give a precise measure.
The most evident example of the Heisenberg principle is the Dirac function δ_0, whose Fourier transform is $2\pi \cdot 1$.

Example 14. If we consider a set of waves, the Fourier transform

$$\hat{f}(\omega) = \int_{\mathbb{R}} e^{-ix\omega} f(x)dx$$

represents the intensity of the wave with oscillation ω.
In particular if we take a Wiener process W_t, we can calculate the derivative, $\dot{W}_t$, in the sense of the distributions. Then, if

$$\mathbb{E}(\dot{W}_t \dot{W}_s) = \delta_{t-s},$$

the Fourier transform $\widehat{\mathbb{E}(\dot{W}_t \dot{W}_s)}$ indicates the intensity of the waves represented by $\mathbb{E}(\dot{W}_t \dot{W}_s)$. In this case it is equal $2\pi \cdot 1$, independent from t and s, that means that each wave appears with the same intensity. For this reason we call $\dot{W}_t$ the *white noise*.
Otherwise, it is called *colored noise*.

Proof (of theorem 14). Let $f \in L^2(\mathbb{R};\mathbb{C})$, $\alpha \in \mathbb{R}$, and $a, b \in \mathbb{R}$, and consider the function

$$(x-a)f(x) + \alpha(f^{'}(x) - ibf(x))$$

If we think at a as the mean of x, the first part $(x-a)f(x)$ corresponds to the function $(x - \langle x \rangle)|f(x)|$ considered in the variance of x. In the same way, the Fourier transform of the second part

$$(f^{'}\widehat{(x) - ib}f(x)) = i\xi \hat{f}(\xi) - ib\hat{f}(\xi) = i(\xi - b)\hat{f}(\xi)$$

corresponds to the function $(\xi - \langle \xi \rangle)|\hat{f}(\xi)|$considered in the variance of ξ.
We want to calculate the quantity

$$\left|(x-a)f(x)+\alpha(f^{'}(x)-ibf(x))\right|^{2}.$$

To do that, call $A := (x-a)f(x) \in \mathbb{C}$ and $B := (f^{'}(x) - ibf(x)) \in \mathbb{C}$. Then

$$|A+\alpha B|^2 = (A+\alpha B)(\bar{A}+\alpha\bar{B}) = |A|^2+\alpha(A\bar{B}+\bar{A}B)+\alpha^2|B|^2$$

Substituting with the value of A and B, we obtain

$$\left|(x-a)f(x)+\alpha(f^{'}(x)-ibf(x))\right|^{2} =$$
$$= (x-a)^2|f(x)|^2+\alpha(x-a)[f(x)\bar{f}^{'}(x)+\bar{f}(x)f^{'}(x)]+\alpha^2|f^{'}(x)-ibf(x)|^2$$

We want to integrate it on $\mathbb{R}$:

$$\int_{\mathbb{R}}(x-a)^2|f(x)|^2dx+\alpha\int_{\mathbb{R}}(x-a)[f(x)\bar{f}^{'}(x)+\bar{f}(x)f^{'}(x)]dx+\alpha^2\int_{\mathbb{R}}\alpha^2|f^{'}(x)-ibf(x)|^2dx$$

Call the first integral A, and notice that $A = \Delta x^2$.
The second integral, B, is

$$\begin{gathered}\int_{\mathbb{R}}(x-a)[f(x)\bar{f}^{'}(x)+\bar{f}(x)f^{'}(x)]dx = \\ = \int_{\mathbb{R}}(x-a)f(x)\bar{f}^{'}(x)+(x-a)\bar{f}(x)f^{'}(x)dx = \\ = \left[(x-a)f\bar{f}\right]_{-\infty}^{+\infty} - \int_{\mathbb{R}}(f+(x-a)f^{'})\bar{f}dx+\int_{\mathbb{R}}(x-a)\bar{f}f^{'}dx = \\ = -\int_{\mathbb{R}}f\bar{f}dx := B\end{gathered}$$

where, at the end, the first term is 0, because $f \in L^2$.
And the third integral, C, is

$$\int_{\mathbb{R}}|f^{'}(x)-ibf(x)|^2dx = \frac{1}{2\pi}\int|\widehat{f'(x)-ibf}(x)|^2d\xi = \frac{1}{2\pi}\int_{\mathbb{R}}i(\xi-b)|\hat{f}(\xi)|^2d\xi = \Delta\xi^2 := C$$

Since $A-\alpha B+\alpha^2C$ is the integral of a modulus it has to be greater or equal 0, that is $A-\alpha B+\alpha^2C \geq 0$. This is a polynomial in the unknown α. Then since $C \geq 0$, we must impose the condition that the determinant is not positive, that is $4AC \geq B^2$: then

$$\Delta x^2\Delta\xi^2 \geq \frac{1}{4}\left(\int_{\mathbb{R}}|f|^2dx\right)^2$$

or also

$$\Delta x\Delta\xi \geq \frac{1}{2}\int_{\mathbb{R}}|f|^2dx = \frac{1}{2}\|f\|^2.$$

We have done all these computations without normalizing the function f. With the normalization we obtain exactly the Heisenberg inequality

$$\Delta x\Delta\xi \geq \frac{1}{2}.$$

2.5 Bernoulli numbers

Let us recall the Dirichlet kernel:

$$\pi D_n(\vartheta) = \frac{\sin(\frac{2n+1}{2}\ \vartheta)}{2\sin\frac{\vartheta}{2}} = \frac{1}{2} + \cos\vartheta + \ldots + \cos(n\vartheta) = \frac{1}{2}\frac{e^{i\vartheta}+e^{-i\vartheta}}{2}\ldots+\frac{e^{in\vartheta}+e^{-in\vartheta}}{2}$$

and consider a function $f \in \mathscr{S}$. Then

$$\int_{\mathbb{R}} f(\vartheta)D_n(\vartheta)d\vartheta = \frac{1}{2\pi}\sum_{k=-n}^{n}\int_{\mathbb{R}} f(\vartheta)e^{-ik\vartheta}d\vartheta = \frac{1}{2\pi}\sum_{k=-n}^{n}\hat{f}(k)$$

We want to study the behaviour of this function when n goes to infinity. For the second term of the equality we have

$$\lim_{n\to\infty}\frac{1}{2\pi}\sum_{k=-n}^{n}\hat{f}(k) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\hat{f}(k).$$

For the first term, we see that there could be problems in the points $\vartheta = 2k\pi$, with $k \in \mathbb{Z}$, indeed

$$\lim_{\vartheta\to 0} D_n(\vartheta) = \frac{1}{2\pi}\lim_{\vartheta\to 0}\frac{\sin(\frac{2n+1}{2}\ \vartheta)}{\frac{2n+1}{2}\ \vartheta}\cdot\frac{\frac{2n+1}{2}\ \vartheta}{\frac{\vartheta}{2}}\cdot\frac{\frac{\vartheta}{2}}{\sin\frac{\vartheta}{2}} = \frac{2n+1}{2\pi}$$

that goes to infinity, as n goes to infinity. Far from this points, the integral of $f(\vartheta)D_n(\vartheta)$ tends to 0, because the positive and the negative parts of $D_n(\vartheta)$ tend to be equal and with opposite sign. Hence, if $\delta << 1$,

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(\vartheta)D_n(\vartheta)d\vartheta = \sum_{k=-\infty}^{\infty}\lim_{n\to\infty}\int_{2k\pi-\delta}^{2k\pi+\delta} f(\vartheta)D_n(\vartheta)d\vartheta.$$

For the continuity of f, this is equal

$$\sum_{k=-\infty}^{\infty} f(2k\pi)\lim_{n\to\infty}\int_{2k\pi-\delta}^{2k\pi+\delta} D_n(\vartheta)d\vartheta.$$

Now, since $\int_{\mathbb{R}} D_n(\vartheta)d\vartheta = 1$ for each n and the integral of the part far from 0 tends to 0, as n goes to infinity, then

$$\lim_{n\to\infty}\int_{\mathbb{R}} f(\vartheta)D_n(\vartheta)d\vartheta = \sum_{k=-\infty}^{\infty} f(2k\pi)$$

Hence we have found the *Poisson formula*

$$\boxed{\sum_{k=-\infty}^{\infty} f(2k\pi) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \hat{f}(k)}$$

Example 15. Let $f = e^{-\alpha|x|}$, with $\alpha > 0$. Then $\hat{f}(\xi)$ is equal

$$\int_{\mathbb{R}} e^{-ix\xi}e^{-\alpha|x|}dx = \int_{\mathbb{R}}(\cos(x\xi)-\sin(x\xi))e^{-\alpha|x|}dx = 2\int_0^{\infty}\cos(x\xi)e^{-\alpha|x|}dx = \frac{2\alpha}{\alpha^2+\xi^2}$$

Then the Poisson formula says

$$\sum_{k=-\infty}^{\infty} e^{-\alpha 2|k|\pi} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{2\alpha}{\alpha^2+\xi^2}$$

Elaborating the first part, we obtain

$$\sum_{k=-\infty}^{\infty} e^{-\alpha 2|k|\pi} = 1+2\sum_{k=1}^{\infty}\left(e^{-\alpha 2\pi}\right)^k = 1+2\frac{e^{-2\pi\alpha}}{1-e^{-2\pi\alpha}} = \coth(\alpha\pi).$$

Then with the Poisson formula we have computed the series

$$\boxed{\frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\frac{2\alpha}{\alpha^2+\xi^2} = \coth(\alpha\pi)}$$

We want to introduce now the *Bernoulli numbers*.
Consider the complex function

$$\varphi(z) = \frac{z}{e^z-1} \qquad \text{for } z\in\mathbb{C}$$

This function is holomorphic in $\mathbb{C}\setminus\{2\pi ki : k\in\mathbb{Z}\}$, because it is the quotient of non zero holomorphic functions. In 0 there is, however, a removable singularity, indeed $\varphi(0) = \lim_{z\to 0}\varphi(z) = 1$. Then the function can be written in 0 with a Taylor series with radious 2π:

$$\varphi(z) = \sum_{k=0}^{\infty}\varphi^{(k)}(0)\frac{z^k}{k!}.$$

We can calculate that $\varphi(0) = 1$ and $\varphi'(0) = -\frac{1}{2}$, then

$$\varphi(z) = 1-\frac{1}{2}z+\varphi''(0)\frac{z^2}{2!}+\dots.$$

Define

$$f(t) := \varphi(z)+\frac{1}{2}z = 1+\varphi''(0)\frac{z^2}{2!}+\varphi'''(0)\frac{z^3}{3!}+\dots.$$

It's easy to see that f is an even function, then all the odd derivatives in 0 are equal 0: $f^{(k)}(0) = 0$ for all k odd. Hence

$$\varphi(z) + \frac{1}{2}z = 1 + B_1\frac{z^2}{2!} + B_2\frac{z^4}{4!} + B_3\frac{z^6}{6!} + \dots.$$

The numbers $B_1, B_2, B_3, \dots$ are called *Bernoulli numbers*.
In some books there is another definition of the Bernoulli numbers: they are the numbers, $\tilde{B}_k$, such that

$$\varphi(t) = \sum_{k=0}^{\infty} \tilde{B}_k \frac{t^k}{k!}$$

Proposition 6. *The Bernoulli numbers satisfy the following equation*

$$\sum_{k=0}^{n-1} \binom{n}{k} \tilde{B}_k = 0 \qquad \forall n \geq 2.$$

Then giving the first term $\tilde{B}_0 = 1$, we can also define the Bernoulli numbers recursively trough this equation.

Proof. We start from the definition of the Bernoulli numbers:

$$\varphi(t) = \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \tilde{B}_k \frac{t^k}{k!} \qquad \forall t \in \mathbb{C} \text{ such that } |t| < 2\pi$$

Then $\tilde{B}_0 = \varphi(0) = 1$. For the other terms consider

$$\varphi(t)(e^t - 1) = t = \sum_{k=0}^{\infty} \tilde{B}_k \frac{t^k}{k!}(e^t - 1) = \left(\sum_{k=0}^{\infty} \tilde{B}_k \frac{t^k}{k!}\right)\left(\sum_{h=1}^{\infty} \frac{t^h}{h!}\right) = \sum_{n=1}^{\infty} \left[\sum_{k+h=n} \tilde{B}_k \frac{1}{k!h!}\right] t^n$$
$$= \sum_{n=1}^{\infty} \left[\sum_{k=0}^{n-1} \binom{n}{k} \tilde{B}_k\right] \frac{t^n}{n!} = t$$

Then the equality holds if and only if

$$\sum_{k=0}^{n-1} \binom{n}{k} \tilde{B}_k = 0 \quad \forall n \geq 2 \qquad \text{and} \qquad \tilde{B}_0 = 1.$$

Let us return to the first notation of Bernoulli numbers

$$\varphi(z) + \frac{1}{2}z = 1 + B_1\frac{z^2}{2!} + B_2\frac{z^4}{4!} + B_3\frac{z^6}{6!} + \dots.$$

We can change notation calling B_{2n} the coefficient corrisponding to the power z^{2n} and thinking B_{2n} as the absolute value of the corresponding Bernoulli number of order $2n$ (this is because the Bernoulli numbers are alternatively positive and negative). Hence we have that

$$\varphi(t) + \frac{t}{2} = \sum_{k=0}^{\infty} (-1)^{k-1} B_{2k} \frac{t^{2k}}{(2k)!}.$$

We can observe also that we have the following identity

$$\frac{t}{e^t - 1} + \frac{t}{2} = \frac{t}{2}\coth\left(\frac{t}{2}\right),$$

hence

$$\boxed{\frac{t}{2}\coth\left(\frac{t}{2}\right) = \sum_{k=0}^{\infty}(-1)^{k-1}B_{2k}\frac{t^{2k}}{(2k)!}}$$

From the Poisson formula we have obtained that

$$\boxed{\frac{\pi\alpha\coth(\pi\alpha) - 1}{2\alpha} = \sum_{k=1}^{\infty}\frac{\alpha}{\alpha^2 + k^2}}$$

We want to use these last computations to calculate the following integral

$$\int_0^{\infty} e^{-kx}\sin(\alpha x)dx.$$

We can note that $e^{-kx}\sin(\alpha x)$ is the imaginary part of the function $e^{(-k+i\alpha)x}$, hence the solution of the previuos integral corresponds to the imaginary part of the function

$$\frac{e^{(-k+i\alpha)x}}{-k+i\alpha},$$

i.e. we have only to compute

$$\Im\left(\frac{e^{(-k+i\alpha)x}}{-k+i\alpha}\right).$$

After some easy computations we have that

$$\Im\left(\frac{e^{(-k+i\alpha)x}}{-k+i\alpha}\right) = \frac{e^{-kx}(-k\sin(\alpha x) - \alpha\cos(\alpha x))}{k^2+\alpha^2},$$

hence

$$\int_0^{\infty} e^{-kx}\sin(\alpha x)dx = \left[\frac{e^{-kx}(-k\sin(\alpha x) - \alpha\cos(\alpha x))}{k^2+\alpha^2}\right]_0^{\infty} = \frac{\alpha}{\alpha^2+k^2}.$$

Summing up we have that

$$\boxed{\int_0^{\infty} e^{-kx}\sin(\alpha x)dx = \frac{\alpha}{\alpha^2+k^2}}$$

We want to use this result to compute another integral that in the traditional way is difficult to solve. Using a previous identity we have that

$$\begin{aligned}\frac{\pi\alpha\coth(\pi\alpha)-1}{2\alpha} &= \sum_{k=1}^{\infty}\frac{\alpha}{\alpha^2+k^2}\\ &= \sum_{k=1}^{\infty}\int_0^{\infty} e^{-kx}\sin(\alpha x)dx\\ &= \int_0^{\infty}\left(\sum_{k=1}^{\infty} e^{-kx}\right)\sin(\alpha x)dx\\ &= \int_0^{\infty}\frac{e^{-x}}{1-e^{-x}}\sin(\alpha x)dx\\ &= \int_0^{\infty}\frac{\sin(\alpha x)}{e^x-1}dx\end{aligned}$$

(we leave as an exercise to the reader to justify why we can invert the integral with the series); hence

$$\boxed{\int_0^{\infty}\frac{\sin(\alpha x)}{e^x-1}dx = \frac{\pi\alpha\coth(\pi\alpha)-1}{2\alpha}}$$

Let us now compare this identity with another one

$$\frac{t}{2}\coth\left(\frac{t}{2}\right) = \sum_{k=0}^{\infty}(-1)^{k-1}B_{2k}\frac{t^{2k}}{(2k)!}.$$

Using $t = 2\pi\alpha$ in this last equation we obtain that

$$\frac{\pi\alpha\coth(\pi\alpha)-1}{2\alpha} = \frac{1}{2\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}B_{2k}\frac{(2\pi\alpha)^{2k}}{(2k)!},$$

hence

$$\int_0^{\infty}\frac{\sin(\alpha x)}{e^x-1}dx = \frac{1}{2\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}B_{2k}\frac{(2\pi\alpha)^{2k}}{(2k)!}. \tag{2.5}$$

Let us recall the Taylor series for $\sin(\alpha x)$,

$$\sin(\alpha x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(\alpha x)^{2k-1}}{(2k-1)!}, \tag{2.6}$$

hence, using (2.6), we have

$$\int_0^{\infty}\frac{\sin(\alpha x)}{e^x-1}dx = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{\alpha^{2k-1}}{(2k-1)!}\int_0^{\infty}\frac{x^{2k-1}}{e^x-1}dx \tag{2.7}$$

(we leave as an exercise to the reader to justify why we can invert the integral with the series). Now, using (2.5) and (2.7), we obtain that

$$\frac{1}{2}\sum_{k=1}^{\infty}(-1)^{k-1}B_{2k}\frac{(2\pi)^{2k}\alpha^{2k-1}}{(2k)!}=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\alpha^{2k-1}}{(2k-1)!}\int_0^{\infty}\frac{x^{2k-1}}{e^x-1}dx;$$

since the coefficients of the same power of α are the same, we have that

$$\frac{1}{2}\frac{(2\pi)^{2k}}{(2k)!}B_{2k}=\frac{1}{(2k-1)!}\int_0^{\infty}\frac{x^{2k-1}}{e^x-1}dx,$$

hence

$$\boxed{\int_0^{\infty}\frac{x^{2k-1}}{e^x-1}dx=\frac{(2\pi)^{2k}}{4k}B_{2k}}$$

Using this formula we can easily compute, for example,

$$\int_0^{\infty}\frac{x}{e^x-1}dx=\frac{\pi^2}{6}\quad\text{and}\quad\int_0^{\infty}\frac{x^3}{e^x-1}dx=\frac{\pi^4}{90}.$$

We want to compute the integral

$$\int_0^{\infty}\frac{x^{2k-1}}{e^x-1}dx$$

in another way. To do this let us recall that the Gamma function on the integer is simply the factiorial, i.e. $\Gamma(2k)=(2k-1)!$, and that, using the Taylor series,

$$\frac{1}{e^x-1}=\sum_{k=1}^{\infty}e^{-kx}.$$

Using these two facts, we obtain that

$$\begin{aligned}\int_0^{\infty}\frac{x^{2k-1}}{e^x-1}dx&=\sum_{n=1}^{\infty}\int_0^{\infty}x^{2k-1}e^{-nx}dx\\&=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\int_0^{\infty}t^{2k-1}e^{-t}dt\\&=\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\Gamma(2k)\\&=(2k-1)!\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\end{aligned}\tag{2.8}$$

(we leave as an exercise to the reader to justify why we can invert the integral with the series). Now, compare (2.8) with the identity in the previous box we obtain the famous *Euler formula*

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^{2k}}=\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}}$$

2.6 Some applications to PDE

The Fourier tranform, and the integral transforms in general, are tools to do other things, for example they are useful to solve every kind of linear problems (for example the heat equation in the whole space, the wave equation in the whole space, the Schrödinger equation, the telegraph equation, the Klein-Gordon equation, the Laplace equation, ...).

2.6.1 The solution of the heat equation

Let us consider the *heat equation*

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = c\dfrac{\partial^2 u(t,x)}{\partial x^2} \\ u(0,x) = f(x) \end{cases}$$

where $c > 0$, $t \geq 0$ and $x \in \mathbb{R}$. To solve this equation we want to take the Fourier transform with respect to x (implicitly we suppose that we are looking for a solution in $\mathscr{S}'$). Using the Fourier transform the initial data becomes $\hat{u}(0,\xi) = \hat{f}(\xi)$ and the equation

$$\frac{\partial \hat{u}}{\partial t} = -c\xi^2 \hat{u},$$

that is an ordinary differential equation that we can solve easily for each ξ, obtaining

$$\hat{u}(t,x) = \hat{f}(\xi)e^{-c\xi^2 t}.$$

Now we have to come back. Let us recall that

$$f(x) \quad \to \quad \hat{f}(\xi),$$

$$g_t(x) = \frac{1}{\sqrt{4\pi ct}} \exp\left(-\frac{x^2}{4ct}\right) \quad \to \quad e^{-c\xi^2 t},$$

and that $\widehat{(g_t * f)} = \hat{g}_t \cdot \hat{f}$, we have a solution of the heat equation given by

$$u(t,x) = \int_{\mathbb{R}} f(y) \frac{1}{\sqrt{4\pi ct}} \exp\left[-\frac{(x-y)^2}{4ct}\right] dy. \tag{2.9}$$

We say a solution and not the solution because a priori we have not a uniqueness theorem. But we can note that when we solve an equation we prove a uniqueness result without proving the exixtence of the solution. To prove the existence we have to verify that the solution we have found soddisfies the equation. Hence with these computations we have proved that, if the solution of the heat equation exists, this is unique in the $\mathscr{S}'$ class functions and has the form (2.9). To prove the existance of the solution we have to check that (2.9) verifies the heat equations.

Remark 9. It is easy to se that $g_t(x)$ is a probability density.

2.6.2 The solution of the Laplace equation

Let us consider the *Laplace equation* in $\mathbb{R}^2$,

$$\begin{cases} \Delta u(x,y) = 0 & \text{on } B(0,R) \\ u(x,y) = f(x,y) & \text{on } \partial B(0,R) \end{cases}$$

i.e. we will find a function $u(x,y) \in C^2(B(0;R))$ such that, if $x^2+y^2 = R^2$, $u(x,y) = f(x,y)$. To solve this equation we transform the problem in polar coordinates using $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$. Hence

$$\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \quad \rightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}.$$

We have also that $u(x,y) = u(r\cos(\vartheta), r\sin(\vartheta)) = u(r,\vartheta)$ where the second u is not the same function as the first u but is the function u transformed with the polar coordinates. Hence we have transformed the Laplace equation in

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} = 0 \\ u(R,\vartheta) = f(\vartheta) \end{cases}$$

We have to solve this new equation.
First of all, we will search a solution of the kind $u(r,\vartheta) = v(r)\varphi(\vartheta)$. What does it happen to the equation if we consider a solution of this kind? This particular choice implies that the problem reduces to

$$\frac{r(rv'(r))'}{v(r)} = -\frac{\varphi''(\vartheta)}{\varphi(\vartheta)},$$

where the first member is indipendent from ϑ and the second member is indipendent from r. This is possible only if

$$\frac{r(rv'(r))'}{v(r)} = -\frac{\varphi''(\vartheta)}{\varphi(\vartheta)} = \lambda,$$

where λ is a constant.
Hence we have reduced our problem to a system of two ordinary differential equations

$$\begin{cases} r(rv'(r))' = \lambda v(r) \\ \varphi''(\vartheta) = -\lambda\varphi(\vartheta) \end{cases}$$

First, let us consider the second equation. If $\lambda < 0$, we can consider $\lambda = -\omega^2$ hence $\varphi'' = \omega^2\varphi$. We will search a solution of kind $\varphi(\vartheta) = e^{\alpha\vartheta}$, hence $\alpha = \pm\omega$. Therefore the solution, if $\lambda = -\omega^2$, will be $\varphi(\vartheta) = Ae^{\omega\vartheta} + Be^{-\omega\vartheta}$. Now, let us observe that the function f must be continuous, hence φ must be periodic. A solution as above is not periodic so we have to consider only the constant λ greater or equal than zero.
If $\lambda = 0$ a general solution for the equation is given by $\varphi(\vartheta) = A\vartheta + B$, but if we impose that φ must be periodic, we will obtain that $A = 0$. Hence, if $\lambda = 0$, the solution is given by $\varphi(\vartheta) = B$.
If $\lambda > 0$, we can consider $\lambda = \omega^2$. Hence the equation becomes $\varphi'' + \omega^2\varphi = 0$, and a general solution is given by $\varphi(\vartheta) = A\cos(\omega\vartheta) + B\sin(\omega\vartheta)$. This solution is periodic of period $T = 2\pi/\omega$; we want that the solution is periodic of period 2π, hence, if we impose this condition, we will obtain that $\omega = k \geq 1$ integer.
Hence $\lambda = k^2$, $k \in \mathbb{Z}$, $k \geq 0$. If we take $\lambda = 0$, we will obtain the previous case. Hence the general solution for the second equation of the system is given by

$$\boxed{\varphi_k(\vartheta) = A_k\cos(k\vartheta) + B_k\sin(k\vartheta) \quad k \in \mathbb{Z},\ \ k \geq 0,\ \ \lambda = k^2}$$

Let us now consider the first equation, i.e. $r(rv'(r))' = \lambda v(r)$. Also in this case we can consider λ as above. After some computations the equation becomes

$$r^2v''(r) + rv'(r) - k^2v(r) = 0.$$

This is a linear ordinary differential equation of second order with non constant coefficients. The solutions of this equation generate a linear space of dimension two, but in general there is not a standard method to solve it and to find two linearly indipendent solutions. Luckily this is a good case. We want to check if there are some solutions of kind $v(r) = r^\alpha$. The equation becomes

$$r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - k^2r^\alpha = 0 \quad \Rightarrow \quad \alpha^2 - k^2 = 0.$$

Hence we have found two linearly indipendent solution $v_1(r) = r^k$ and $v_2(r) = r^{-k}$. Therefore the general solution in this case is given by $v(r) = A_kr^k + B_kr^{-k}$, where k is an integer greater or equal than one. Let us observe that this solution v has a singularity in zero. Because we will search a solution of class C^2, we have to impose that $B = 0$ to cancel the singularity. Hence the solution becomes $v(r) = A_kr^k$, $k \geq 1$ integer.
Let us now consider the case $k = 0$. The equation becomes $r^2v''(r) + rv'(r) = 0$. If we put $w = v'$, we obtain that $rw' + w = 0$. A solution of this second equation is given by $w = A/r$, hence a solution of the equation in v is given by $v(r) = A\log(r) + C$. This solution has a singularity in zero, hence, to have a solution of class C^2, we need $A = 0$. Hence the solution, if $k = 0$, is $v(r) = C$ that is the same of the previous case (imposing $k = 0$).
Hence, the general solution for the first equation of the system is given by

$$\boxed{v_k(r) = C_kr^k \quad k \in \mathbb{Z},\ \ k \geq 0,\ \ \lambda = k^2}$$

Summing up, if we will look for a solution for the system of kind $u(r,\vartheta) = v(r)\varphi(\vartheta)$, we obtain

$$\boxed{u_k(r,\vartheta) = r^k \left(A_k\cos(k\vartheta) + B_k\sin(k\vartheta)\right) \quad k \in \mathbb{Z},\ \ k \geq 0\ \ r < R}$$

Let us now impose the condition on the boundary, that is for $r = R$. Then the found solution takes on the boundary the value

$$u(R,\vartheta) = R^k(a_k\cos(k\vartheta) + b_k\sin(k\vartheta))$$

that is a very restrictive condition. However, try to write the condition f as a Fourier series:

$$f(\vartheta) = \frac{A_0}{2} + \sum_{k=1}^{\infty}(A_k\cos(k\vartheta) + B_k\sin(k\vartheta))$$

To do that we have to impose some stronger hypotheses to this f. If we take $f \in L^2(0,2\pi)$, we get only a convergence in L^2 (not pointwise). To have a pointwise convergence, we have studied in the previous chapter that there are two possible conditions:

HP1) f continuous, periodic and $f(\vartheta) = c + \int_0^{\vartheta} g(t)dt$ whit $g \in L^2(0,2\pi)$
HP2) $f \in \mathscr{C}^{\alpha}(0,2\pi)$

Now observe that $u(r,\vartheta) := \sum_{k=0}^{\infty} u_k(r,\vartheta)$ satisfies $\Delta u = 0$, because the Laplace operator is linear and $\Delta u = \sum_{k=0}^{\infty} \Delta u_k(t,x) = 0$. Impose now the condition on the boundary, that is

$$u(r,\vartheta) = \sum_{k=0}^{\infty}[R^k a_k\cos(k\vartheta) + R^k b_k\sin(k\vartheta)] = f(\vartheta)$$

so, from the Fourier series of f we obtain that the coefficients of the solution are

$$a_0 = \frac{A_0}{2} \qquad a_k = \frac{A_k}{R^k} \qquad b_k = \frac{B_k}{R^k}$$

and the solution becomes

$$\boxed{u(r,\vartheta) = \frac{A_0}{2} + \sum_{k=1}^{\infty}\left(\frac{r^k}{R^k}A_k\cos(k\vartheta) + \frac{r^k}{R^k}B_k\sin(k\vartheta)\right)}$$

under some strong hypotheses: HP1 or HP2.
It is left as an exercise to verify that $u(r,\vartheta)$ is really a solution of the Laplace problem. Then with this procedure we have shown under these strong hypotheses, also the uniqueness of the solution.
Writing the expression of a_k and b_k in the formula of $u(r,\vartheta)$, we obtain

$$\begin{aligned} u(r,\vartheta) &= \frac{1}{2\pi}\int_0^{2\pi} f(s)ds + \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{r^k}{R^k}\int_0^{2\pi} f(s)\{\cos(k\vartheta)\cos(ks)+\sin(k\vartheta)\sin(ks)\}ds \\ &= \frac{1}{\pi}\int_0^{2\pi} f(s)[\frac{1}{2}+\sum_{k=1}^{\infty}\frac{r^k}{R^k}\cos(k(s-\vartheta))]ds \end{aligned}$$

We need to have a good expression for $u(r,\vartheta)$, so we have to simplify this integral: let us call $\frac{r}{R}=\rho<1$ and $s-\vartheta=\alpha$. Then

$$\frac{1}{2}+\sum_{k=1}^{\infty}\rho^k\cos(k\alpha)=\frac{1}{2}+\Re\left[\sum_{k=1}^{\infty}(\rho e^{i\alpha})^k\right]=\frac{1}{2}+\Re\left[\frac{\rho e^{i\alpha}}{1-\rho e^{i\alpha}}\right]=\frac{1}{2}+\rho\,\Re\left[\frac{e^{i\alpha}}{1-\rho e^{i\alpha}}\right]$$

Now considering only the complex fraction

$$\begin{aligned} \Re\left[\frac{e^{i\alpha}}{1-\rho e^{i\alpha}}\right] &= \Re\left[\frac{\cos(\alpha)+i\sin(\alpha)}{(1-\rho\cos(\alpha))-i\rho\sin(\alpha)}\right] \\ &= \Re\left[\frac{\cos(\alpha)+i\sin(\alpha)(1-\rho\cos(\alpha))+i\rho\sin(\alpha)}{(1-\rho\cos(\alpha))^2+\rho^2\sin^2(\alpha)}\right] \\ &= \frac{\cos(\alpha)(1-\rho\cos(\alpha))-\rho\sin^2(\alpha)}{1-2\rho\cos(\alpha)+\rho^2} \\ &= \frac{\cos(\alpha)-\rho}{1-2\rho\cos(\alpha)+\rho^2} \end{aligned}$$

Hence, replacing it in the last formula

$$\frac{1}{2}+\sum_{k=1}^{\infty}\rho^k\cos(k\alpha)=\frac{1-\rho^2}{2(1-2\rho\cos(\alpha)+\rho^2)}$$

Then

$$u(r,\vartheta)=\frac{1}{2\pi}\int_0^{2\pi} f(s)\frac{R^2-r^2}{R^2-2Rr\cos(s-\vartheta)+r^2}ds$$

Define the *Poisson kernel*

$$\boxed{K(r,\alpha)=\frac{1}{2\pi}\frac{R^2-r^2}{R^2-2Rr\cos(s-\vartheta)+r^2}}$$

hence the solution is a convolution product:

$$u(r,\vartheta)=\int_0^{2\pi} f(s)K(r,s-\vartheta)ds$$

Notice that this last formula has a sense also when f is only continuous. So we can choose it as a candidate for the solution also of the system whitout the stronger conditions that we have imposed to f.

Exercise 12 *Prove that the function $u(r,\vartheta)$ solves the equation $\Delta u = 0$.*
(Hint: *Use the polar coordinates and notice that the derivative is with respect to the coordinate r, so the Laplace operator acts only on $K(r,\alpha)$ and not on f.)*
(Warning: *Since $\Delta K \notin L^1$, the integral doesn't exists in the usual meaning. It is to be studied as a singular integral.)*

Remark 10 (Properties of the Poisson Kernel). $K(r,\alpha) \geq 0$, indeed the numerator $R^2 - r^2$ is not negative, because $r \leq R$, and the denominator comes from $(R - r\cos(\alpha))^2 + (r\sin(\alpha))^2 \geq 0$.
Moreover $\int_0^{2\pi} K(r,\alpha)d\alpha = 1$, because $K(r,\alpha) = \frac{1}{\pi}\left[\frac{1}{2} + \sum_{k=1}^{\infty} \frac{r^k}{R^k}\cos(k\alpha)\right]$. This is not by chance, indeed $K(r,\alpha)$ is a probability density.

Remark 11. When r tends to R, the kernel $K(r,\alpha)$ tends to the Dirac-delta $\delta_0(\alpha)$. Hence

$$\lim_{r\to R} u(r,\vartheta) = \lim_{r\to R} \int_0^{2\pi} K(r,\vartheta - s) f(s)ds = \int_0^{2\pi} \delta_\vartheta(s) f(s)ds = f(\vartheta)$$

then the condition on the boundary is satisfied.

Let us now consider the Laplace equation on the half upper plane G

$$\begin{cases} \Delta u(x_1,x_2) = 0 & \text{on } G \\ u(x_1,0) = f(x_1) & \text{on } \partial G \end{cases}$$

where $G := \left\{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\right\}$.
To solve it we try taking the Fourier trasform of $\Delta u = 0$ with respect to the variable x_1:

$$0 = \mathscr{F}(\Delta u) = \mathscr{F}\left(\frac{\partial^2 u}{\partial x_1^2}\right) + \mathscr{F}\left(\frac{\partial^2 u}{\partial x_2^2}\right) = -\xi^2(\mathscr{F}u)(\xi,x_2) + \frac{\partial^2}{\partial x_2^2}(\mathscr{F}u)(\xi,x_2)$$

In this way, if for each fixed ξ, we call $(\mathscr{F}u)(\xi,x_2) = \varphi(x_2)$, we have obtained a linear differential equation

$$\begin{cases} \varphi''(\xi,x_2) = \xi^2\varphi(\xi,x_2) \\ \varphi(\xi,0) = \hat{f}(\xi) \end{cases}$$

that has the solution

$$\varphi(\xi,x_2) = A_\xi e^{|\xi|x_2} + B_\xi e^{-|\xi|x_2}$$

Now recall that we are considering the Fourier transform of u, that is at least in the set $\mathscr{S}'$. The function $e^{|\xi|x_2}$ however is not in $\mathscr{S}'$, then we must force $A_\xi = 0$. Then we have found the solution

$$\hat{u}(\xi,x_2) = \hat{f}(\xi)e^{-x_2|\xi|}$$

Now he have to come back to the function u. Observe that $\mathscr{F}^{-1}(e^{-x_2|\xi|}) = \frac{1}{\pi}\frac{x_1}{x_1^2+x_2^2}$ and remember the formula of the inverse Fourier transform of a product. Then

$$\boxed{u(x_1,x_2) = \frac{1}{\pi}\int_{\mathbb{R}} f(y)\frac{x_1}{x_1^2+(x_2-y)^2}dy}$$

Exercise 13 *Find a conformal transform $\Phi : \mathbb{C} \to \mathbb{C}$ that sends the unitar sphere in the upper half plane.*

Chapter 3
Laplace Transform

3.1 Recalls on Banach spaces

We want to study the Fourier transform $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ from the point of view of the Banach spaces.
A Banach space is a normed vector space that is also complete. Some examples of Banach spaces are L^p, C^α with $0 < \alpha \leq 1$, C, $W^{x,p}$ the Sobolev spaces. Moreover, if the norm is defined by a scalar product, the space is a Hilbert space. In particular L^2 is a Hilbert space.
If X and Y are Banach spaces, a map $T : X \to Y$ is linear if for each $x, y \in X$, $\lambda \in \mathbb{R}$, $T(x + \lambda y) = T(x) + \lambda T(y)$ and the Fourier transform is an example of linear operator.
There are two classes of linear operators:

(i) the bounded operators: T is bounded if $\|Tx\| \leq M \|x\|$. In this case we define the norm of T as
$$\|T\| := \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\|$$
(ii) the unbunded operators.

Exercise 14 *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a bounded linear operator. Endow $\mathbb{R}^n$ with the euclidean norm $\|x\|^2 = \sum_{i=1}^n x_i^2$. What is the norm of A?*

We want to calculate now the norm of the Fourier transform. For this goal we introduce another very similar operator. Recall that $(\mathscr{F} f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$ and define the operator
$$(Uf)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx$$
that is $U = \frac{1}{\sqrt{2\pi}} \mathscr{F}$. In particular this U conserves the scalar product:
$$\langle Uf, Ug \rangle = \langle f, g \rangle$$

U is also an *unitar operator*, that is $U^* = U^{-1}$, indeed

$$(U^{-1}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(x)dx$$

then $||U|| = 1$.
There are five fundamental theorems that we have to remember about the Banach spaces and continuous linear operators. We cite these five theorems (for a proof see for example [R86] chapter 5 or [B83] chapters 1 and 2).

Theorem 15 (Baire). *Let (X,d) be a complete metric space. Suppose that there exists a sequence $\{X_n\}_n$ of closed subsets of X such that $X = \bigcup_{n=0}^{\infty} X_n$. Then there exists n_0 for which int $\left(X_{n_0}\right) \neq \emptyset$.*

Theorem 16 (Hahn-Banach). *If M is a subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X so that $\|F\| = \|f\|$.*

Theorem 17 (Banach-Steinhaus). *Suppose X is a Banach space, Y is a normed linear space and $\{T_i\}$ is a collection of bounded linear transformations of X into Y, where i ranges over some index set I. Suppose that for each $x \in X$ there exists a constant $c(x) \in (0,\infty)$ such that*

$$\sup_{i \in I} \|T_i(x)\| \leq c(x).$$

Then there exists $L > 0$ such that $\|T_i(x)\| \leq L\|x\|$ for each $x \in X$ and $i \in I$.

Theorem 18 (Open Mapping). *Let X and Y be Banach spaces. If $T : X \to Y$ linear is surjective then T is open, i.e., $T(U)$ is an open subset of Y whenever U is an open subset of X.*

Theorem 19 (Closed Graph). *Let X and Y be Banach spaces. If If $T : X \to Y$ is linear and its graph*

$$G(T) := \{(x, T(x)) : \ x \in X\}$$

is closed in $X \times Y$, then T is continuous.

If a linear operator T is unbounded, then T isn't defined on the whole space X, but only on a subspace $D_T \subset X$, so $T : D_T \subset X \to X$. An example of unbounded linear mapping is the derivative and in this case the subspace D_T is $D_T = C_c^1(\mathbb{R}) \subset L^2(\mathbb{R})$, the space of C^1 function with compact support.

Exercise 15 *The derivative T is an unbounded linear mapping. Find a sequence $(\varphi_n)_n \subset C_c^1(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} |\varphi_n'|^2 dx \to \infty \quad \textit{for } n \to \infty \qquad \textit{and} \qquad \int_{\mathbb{R}} |\varphi_n|^2 dx \leq 1 \ \ \forall n.$$

The bounded operators are relatively simple to study. The unbounded operators are more difficult and the problems of PDE regard the unbunded operators. This is the reason to introduce the Sobolev spaces.
Let g be a measurable function and define a linear operator from the space $L^2(\mathbb{R})$. For $f \in L^2(\mathbb{R})$ define

$$T_g f := g(x) f(x)$$

If $g \in L^\infty$ and $f \in L^2$, then $fg \in L^2$, indeed

$$\int (gf)^2 dx \leq M \int f^2 dx < \infty$$

where $|g(x)|^2 \leq M$ for each $x \in \mathbb{R}$. In particular this proves also that

$$||T_g f||_{L^2} \leq ||g||_\infty ||f||_{L^2} \quad \text{and} \quad ||T_g|| \leq ||g||_\infty$$

then if $g \in L^\infty$, $T_g : L^2 \to L^2$ is bounded.
If we take $g \notin L^\infty$ unbounded, it doesn't hold that $T_g(L^2) \subset L^2$ and we have to define the function T_g on a smaller space, $D_{T_g} = \{f \in L^2 : gf \in L^2\}$.
For example consider $g(x) = x \notin L^\infty$. If

$$f(x) = \begin{cases} \frac{1}{x} & \text{if } |x| \geq 1 \\ 0 & \text{if } |x| < 1 \end{cases}$$

then $T(f) = fg \notin L^2$. But all the C^∞-functions with compact support are in D_T. So we can define T on the space $D_T^{'} = C_c^\infty \subset D_T$, $T : D_T^{'} \to L^2$.

Exercise 16 *Let $\{f_n\}_n \subset D_T$ and suppose that*

$$\begin{cases} f_n \to f & \textit{in } L^2 \\ T f_n = x f_n(x) \to h & \textit{in } L^2 \end{cases}$$

Prove that $f \in D_T$ and $h = T(f)$.
This means that $T(f) = xf$ defined on D_T is a closed operator.

Notice that if T is bounded, then T is a closed operator.

3.2 Laplace Transform

Let us recall that the Fourier transform is something like $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$, where $\mathscr{S}'$ is the dual space of $\mathscr{S}$. This dual space is called *tempered distribution* and we have that, if $T \in \mathscr{S}'$,

$$\langle \mathscr{F}(T), \varphi \rangle = \langle T, \mathscr{F}(\varphi) \rangle \quad \forall\, \varphi \in \mathscr{S}.$$

Let us also recall that $C_c^\infty \subset \mathscr{S} \subset L^2 \subset \mathscr{S}' \subset \mathscr{D}'$, where $\mathscr{D}'$ is the dual space of C_c^∞ (to define the dual of C_c^∞ we have to put on it a topology τ: the couple $(C_c^\infty, \tau) = \mathscr{D}$

is called *distruibution space*).
It easy to see, for example, that T_f where $f(x) = x$ is an operator in $\mathscr{S}'$ in fact the integral

$$\int_{\mathbb{R}} x\varphi(x)dx$$

has a sense for any φ in $\mathscr{S}$. To see this statement we have to observe that $\varphi \in \mathscr{S}$ so $\varphi(x)(1+|x|^n) \leq C_n$ for all n, hence

$$\int_{\mathbb{R}} x\varphi(x)dx = \int_{\mathbb{R}} \frac{x}{1+|x|^4}(1+|x|^4)\varphi(x)dx \leq \int_{\mathbb{R}} C_4 \frac{x}{1+|x|^4}dx < +\infty.$$

Are there some easy function not in $\mathscr{S}'$? For example the function e^x is not in $\mathscr{S}'$ bacause we cannot find any polynomial for which the integral

$$\int_{\mathbb{R}} e^x\varphi(x)dx$$

has a sense. Hence we can not define the Fourier transform of the exponential funtion. However, $e^x \in \mathscr{D}'$ because in this case φ is in C_c^∞. Hence the integral exists because of the compact support of the function φ.
Because is interesting to use e^x, a frequent function in analysis, it is convenient introduce the *Laplace Transform*.
To define the Laplace transform we have a price to pay: we can not define the transform on the whole space but only on the positive half plane $\mathbb{R}^+ = \{t \geq 0\}$. Let us suppose that $|f(t)| \leq Me^{\omega t}$. Let us define the Laplace transform as

$$\widetilde{f}(p) = \int_0^\infty e^{-pt}f(t)dt \quad \text{for } p \in \mathbb{C}.$$

Because of $p \in \mathbb{C}$, $p = x + iy$ hence

$$\widetilde{f}(p) = \int_0^\infty e^{-pt}f(t)dt = \int_0^\infty e^{-xt}f(t)e^{-iyt}dt.$$

We can extend f to 0 out of $\mathbb{R}^+$ hence

$$\widetilde{f}(p) = \int_{-\infty}^{+\infty} e^{-xt}f(t)e^{-iyt}dt.$$

This is the Fourier transform of the function $e^{-xt}f(t)$. $e^{-xt}f(t)$ is the weight to compensate the weight of the exponential function because $|f(t)| \leq Me^{\omega t}$. For which x the integral above has a sense? For $x > \omega$ because $|e^{-xt}f(t)| \leq Me^{t(\omega - x)}$: if $x > \omega$ we have that $\omega - x$ is negative and $e^{-xt}f(t)$ is integrable. Hencer x must be large enough to compensate the growth of the function $f(t)$.
Hence $\widetilde{f}(p)$ is a function of p with $\Re(p) > \omega$.

Definition 10. If $f(t) \in L^1_{loc}(\mathbb{R}^+)$ and has an exponential growth, i.e., $|f(t)| \leq Me^{\omega t}$, we define the *Laplace Transform* as

$$\widetilde{f}(p) = \int_0^\infty e^{-pt} f(t)dt \quad \forall\, p:\ \Re(p) > \omega.$$

Fig. 3.1 Pierre Simon Laplace (1749-1827)

The map $f(t) \to \widetilde{f}(p)$ is linear i.e. $(\widetilde{\lambda f + g}) = \lambda \widetilde{f} + \widetilde{g}$, hence the Laplace transform is a linear operator.

Example 16. Let us consider the function $f_\alpha(t) = e^{\alpha t}$ with $\alpha > 0$. This function suddisfies all the hypotesis hence we can compute its Laplace transform. Let us suppose $\Re(p) > \alpha$. So

$$\widetilde{f}(p) = \int_0^\infty e^{t(\alpha - p)} dt = \left[\frac{e^{t(\alpha-p)}}{\alpha - p} \right]_0^\infty = \frac{1}{p - \alpha}.$$

The function $p \to \frac{1}{p-\alpha}$ is an holomorphic function on the half plane $\left\{(x,y) \in \mathbb{R}^2 :\ x > \alpha\right\}$. However the function has sense also out of that half plane. Hence we can consider an holomorphic extension of the function out of the half space except for the point $(\alpha, 0)$. This is important to invert the Laplace transform.

We want now prove this in general:

Proposition 7. *$\widetilde{f}$ is an holomorphic function on the half-space $\{p \in \mathbb{C} :\ \Re(p) > \omega\}$.*

Proof. We have to prove that $\widetilde{f}$ is derivable in the complex sense.
Let $p \in \{p \in \mathbb{C} :\ \Re(p) > \omega\}$, let us consider a circular neighbourhood centred in p with radius ρ, $B(p,\rho)$, and let Δp an increment in $B(p,\rho)$. Let us consider

$$\frac{\widetilde{f}(p + \Delta p) - \widetilde{p}}{\Delta p} = \int_0^\infty \frac{e^{-\Delta p t} - 1}{\Delta p} e^{-pt} f(t) dt.$$

We have to prove that there exists the limit of the previous expression when $|\Delta p| \to 0$.
To do this we want to use the dominated convergence theorem. We already know that

$$\lim_{|\Delta p|\to 0} \frac{e^{-\Delta pt}-1}{\Delta p} = -t,$$

because $h(t) = e^{-zt}$ is an holomorphic function; hence

$$\lim_{|\Delta p|\to 0} \frac{e^{-\Delta pt}-1}{\Delta p} e^{-pt} f(t) = -te^{-pt} f(t).$$

So, if there exists a function $g(t) \in L^1(\mathbb{R}^+)$, indipendent from Δp, such that

$$\left| \frac{e^{-\Delta pt}-1}{\Delta p} e^{-pt} f(t) \right| \leq g(t), \tag{3.1}$$

we can apply the dominated convergence theorem to state that

$$\frac{\widetilde{f}(p+\Delta p)-\widetilde{p}}{\Delta p} \overset{|\Delta p|\to 0}{\to} -\int_0^\infty e^{-pt} t f(t) dt < +\infty,$$

because

$$-\int_0^\infty e^{-pt} t f(t) dt \leq -\int_0^\infty e^{-\Re(p)t} t \,|f(t)|\, dt \leq -\int_0^\infty Mte^{-(\Re(p)-\omega)t} dt < +\infty.$$

Hence, it remains to prove that there exists $g \in L^1$, indipendent of Δp, for which (3.1) holds. Let us start observing that

$$\frac{e^{-zt}-1}{z} = -\int_0^t e^{-zs} ds.$$

Hence

$$\left| \frac{e^{-zt}-1}{z} \right| \leq \int_0^t \left|e^{-zs}\right| ds = \int_0^t e^{-\Re(z)s} ds.$$

If $\Re(z) > 0$, we have that $e^{-\Re(z)s} \leq 1$, and, if $\Re(z) < 0$, we have that $e^{-\Re(z)s} \leq e^{\rho s}$. In each case we can consider $e^{-\Re(z)s} \leq e^{\rho s}$. Hence

$$\left| \frac{e^{-zt}-1}{z} \right| \leq \int_0^t e^{\rho s} ds \leq \int_0^t e^{\rho t} ds = te^{\rho t}.$$

Hence

$$\left| \frac{e^{-\Delta pt}-1}{\Delta p} e^{-pt} f(t) \right| = \left| \frac{e^{-\Delta pt}-1}{\Delta p} \right| e^{-\Re(p)t} \,|f(t)| \leq te^{\rho t} e^{-\Re(p)t} \,|f(t)| \leq Me^{(\omega-\Re(p)+\rho)t},$$

that is integrable on $\mathbb{R}^+$ for a suitable value of ρ (we have to choose $\rho < \Re(p) - \omega$). Hence if we take $g(t) = Me^{(\omega - \Re(p) + \rho)t}$, with $\rho < \Re(p) - \omega$, $g(t)$ is integrable, indipendent from Δp and verifies the condition (3.1).

3.2.1 An application: the heat equation

Let us now consider an application: we want to solve the *heat equation* using the Laplace transform. Let us consider the heat equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = c\dfrac{\partial^2 u(t,x)}{\partial x^2} \\ u(0,x) = f(x) \end{cases}$$

where $c > 0$, $t \geq 0$ and $x \in \mathbb{R}$.
We want to solve this problem in $L^2(\mathbb{R})$, i.e., we want to find a function $u \in L^2(\mathbb{R})$ that satisfies the heat equation with $f \in L^2(\mathbb{R})$. To solve it we can use the Laplace transform with respect to the variable t (and not with respect to x like in the case of the Fourier transform).

Remark 12. $f'(t) \to \widetilde{f}'(p)$, where

$$\widetilde{f}'(p) = \int_0^\infty e^{-pt} f'(t) dt = \left[e^{-pt} f(t)\right]_0^\infty + p\int_0^\infty e^{-pt} f(t) dt = p\widetilde{f}(p) - f(0).$$

Hence, we have

$$\widetilde{\left(\frac{\partial u}{\partial t}\right)} = p\widetilde{u}(p,x) - u(0,x) = c\frac{\partial^2}{\partial x^2}\widetilde{u}(p,x).$$

Now, using the initial data, we obtain that

$$\left(p - c\frac{\partial^2}{\partial x^2}\right)\widetilde{u} = f,$$

that is an elliptic problem.
Hence the Laplace transform transforms the heat equation that is a parabolic problem in a family of elliptic problems. (The Laplace transform is the base of the semigroup theory that is a theory to treat parabolic and hyperbolic problems of this kind).
Hence

$$\widetilde{u} = \left(p - c\frac{\partial^2}{\partial x^2}\right)^{-1} f,$$

this writing is only formal: the sense is that, given f and given the operator, we have to find $\widetilde{u}$.
The problem becomes very simple in one dimension: let us call $v = \widetilde{u}$; the problem becomes $pv - cv'' = f$ with $x \in \mathbb{R}$, that is a second order ODE with constant coeffi-

cients and $p \in \mathbb{C}$, that we want to solve in $L^2(\mathbb{R})$. To solve it we can use the Fourier transform with respect to x obtaining $p\hat{v} + c\xi^2\hat{v} = \hat{f}$. Hence we have

$$\hat{v}(\xi) = \frac{\hat{f}(\xi)}{p + c\xi^2} \quad \text{in } L^2(\mathbb{R}).$$

For which $p \in \mathbb{C}$ can we solve this equation in $L^2(\mathbb{R})$? If $p = -c\xi_0^2$ with $\xi_0 = \sqrt{|p/c|}$ we have a singularity, hence, in general, we have a singularity if $\Im(p) = 0$ and $p < 0$. For the other cases there is no singularity, hence we could solve the equation if $\hat{v}(\xi)$ is in $L^2(\mathbb{R})$.
We want prove that, if the hypotesies $\Im(p) = 0$ and $p < 0$ are not soddisfy, $\hat{v}(\xi)$ is in $L^2(\mathbb{R})$.
If $p \in \mathbb{C}$ is such that $\Re(p) > 0$, we have that

$$\left|p + c\xi^2\right| = \left|p - (-c\xi^2)\right| \geq |p| \quad \Rightarrow \quad \left|\frac{1}{p + c\xi^2}\right| \leq \frac{1}{|p|} \quad \Rightarrow \quad \hat{v}(\xi) \leq \frac{\left|\hat{f}(\xi)\right|}{|p|},$$

where $|p|$ represents the minimum of the distance between $p \in \mathbb{C}$ and the set of real negative numbers $\{z \in \mathbb{C} : \Im(z) = 0 \text{ and } z < 0\}$.
If $p \in \mathbb{C}$ is such that $\Re(p) \leq 0$, we have that

$$\left|p + c\xi^2\right| = \left|p - (-c\xi^2)\right| \geq |\Im(p)| \quad \Rightarrow \quad \left|\frac{1}{p + c\xi^2}\right| \leq \frac{1}{|\Im(p)|} \quad \Rightarrow \quad \hat{v}(\xi) \leq \frac{\left|\hat{f}(\xi)\right|}{|\Im(p)|},$$

where $|\Im(p)|$ represents the minimum of the distance between $p \in \mathbb{C}$ and the set of real negative numbers $\{z \in \mathbb{C} : \Im(z) = 0 \text{ and } z < 0\}$.
In every case we have that, if $f \in L^2(\mathbb{R})$, $\hat{v} \in L^2(R)$, hence we can apply the inverse Fourier transform to solve the equation and find the solution of the heat equation

$$v(x) = \mathscr{F}^{-1}\left(\frac{\hat{f}(\xi)}{p + c\xi^2}\right) \quad \text{if } p \notin \{z \in \mathbb{C} : \Im(z) = 0 \text{ and } z < 0\}.$$

Hence we have the solution on the whole space except $\{z \in \mathbb{C} : \Im(z) = 0 \text{ and } z < 0\}$.
Hence we can apply the inverse Laplace transform to find $u(t,x)$ because this is defined on the half space $\{z \in \mathbb{C} : \Re(z) > \omega\}$ that is conteined in the set of definition of $v(x) = \widetilde{u}(t,x)$.
It is interesting to note how we do not find an explicit solution of the problem, but we use the properties of the problem itself to prove that the solution exists and to find an implicit expression of it.

3.2.2 An application: the Abel equation

Let us consider the following problem:

$$g(t) = \int_0^t \frac{f(s)}{(t-s)^\alpha} ds \quad 0 < \alpha < 1,$$

where $f, g \in L^1_{loc}(\mathbb{R}^+)$ extended to 0 for $t < 0$ such that $|g(t)|, |f(t)| \leq Me^{\omega t}$ for $t \geq t_0$ for a suitable value $t_0 \in \mathbb{R}^+$. Moreover, g is given, hence the unknown of the problem is the function f. This is a particular case of the *linear Volterra integral equation of convolution type* called *Abel equation*.
First of all, let us note that, if $h_\alpha(t) = \frac{1}{t^\alpha}$, the problem could be rewritten as $g(t) = (f * h_\alpha)$. Now the idea is to apply the Laplace transform on both members.
We know the convolution product between two L^1-functions and not between functions in $L^1_{loc}(\mathbb{R}^+)$. Hence, we have to study if it is possible to define a convolution product between two function in L^1_{loc}. If $f, g \in L^1_{loc}$,

$$(f * g)(t) = \int_{\mathbb{R}} f(t-s)g(s)ds.$$

If this integral has a meaning, we have defined a convolution product on L^1_{loc}. But

$$\int_{\mathbb{R}} f(t-s)g(s)ds = \int_0^t f(t-s)g(s)ds,$$

because $g(s) = 0$ for $s < 0$ and $f(t-s) = 0$ for $s > t$; and this integral have always a meaning because f and g are in L^1_{loc}. Hence, if $f, g \in L^1_{loc}$,

$$(f * g)(t) = \int_0^t f(t-s)g(s)ds.$$

Proposition 8. *If $f, g \in L^1_{loc}$ such that $|g(t)|, |f(t)| \leq Me^{\omega t}$ for $t \geq t_0$ for a suitable value $t_0 \in \mathbb{R}^+$,*

$$\widetilde{(f * g)}(p) = \widetilde{f}(p) \cdot \widetilde{g}(p).$$

Proof.

$$\begin{aligned}
\widetilde{(f*g)}(p) &= \int_0^\infty e^{-pt}(f*g)(t)dt \\
&= \int_0^\infty e^{-pt}\left(\int_0^t f(t-s)g(s)ds\right)dt \\
&= \int_0^\infty g(s)\left(\int_s^\infty e^{-pt}f(t-s)dt\right)ds \\
&= \int_0^\infty e^{-ps}g(s)\left(\int_s^\infty e^{-p(t-s)}f(t-s)dt\right)ds \\
&= \int_0^\infty e^{-ps}g(s)\left(\int_0^\infty e^{-pz}f(z)dz\right)ds \\
&= \left(\int_0^\infty e^{-pz}f(z)dz\right)\left(\int_0^\infty e^{-ps}g(s)ds\right) \\
&= \widetilde{f}(p)\cdot\widetilde{g}(p)
\end{aligned}$$

Returning to the Abel equation, we have

$$g(t) = (f*h_\alpha)(t) \quad\Rightarrow\quad \widetilde{g}(p) = \widetilde{f}(p)\cdot\widetilde{h_\alpha}(p) \quad\Rightarrow\quad \widetilde{f}(p) = \frac{\widetilde{g}(p)}{\widetilde{h_\alpha}(p)}.$$

Hence, we have only to compute the Laplace transform of $h_\alpha(t)$ and to come back. Can we do the Laplace transform of $h_\alpha(t)$? It is trivial that $|h_\alpha(t)| \leq Me^{\omega t}$ for $t > t_0$ for a suitable value of t_0. Moreover, $h_\alpha(t) \in L^1_{loc}(\mathbb{R}^+)$ because

$$\int_0^T \frac{1}{t^\alpha}dt < +\infty \quad \forall\, T > 0.$$

Hence we can do the Laplace transform and

$$\begin{aligned}
\widetilde{h_\alpha}(p) &= \int_0^\infty \frac{e^{-pt}}{t^\alpha}dt = p^{\alpha-1}\int_0^\infty \frac{e^{-(pt)}}{(pt)^\alpha}d(pt) \\
&= \frac{1}{p^{1-\alpha}}\int_0^\infty e^{-s}s^{-\alpha}ds = \frac{1}{p^{1-\alpha}}\Gamma(1-\alpha),
\end{aligned}$$

where $\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt$ for $x > 0$ is the *gamma function*.
The computations above are rigorous only if p is a real positive number. If $p \in \mathbb{C}$, the result is still true: if $p \in \mathbb{C}$, $s = pt \in \mathbb{C}$ the integral is no longer on $[0, +\infty)$ but it should computed along another path in the complex plane.
Moreover, there is another point: if $p \in \mathbb{C}$ what is the meaning of $p^{1-\alpha}$? Which value of $p^{1-\alpha}$ choose? We can choose to take it so that $p^{1-\alpha}$ is an holomorphic function on a ball centred in 1 with radius 1, but, if we extend this choose we arrive at the end of the circle to have a step, a discontinuity. To solve this problem we have to consider the complex plane without the real negative axis, i.e.

$\mathbb{C} \setminus \{z \in \mathbb{C} : \Im(z) = 0 \text{ and } \Re(z) \geq 0\}$, and than choose one of the value, generally which one with positive real part (because the two values can be different).
Hence

$$\begin{aligned}\widetilde{f}(p) &= \frac{p^{1-\alpha}}{\Gamma(1-\alpha)}\widetilde{g}(p) = \frac{1}{\Gamma(1-\alpha)}p\widetilde{g}(p)\frac{1}{p^{\alpha}} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}p\widetilde{g}(p)\frac{\Gamma(\alpha)}{p^{\alpha}} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left[p\widetilde{g}(p) - g(0) + g(0)\right]\frac{\Gamma(\alpha)}{p^{\alpha}} = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left[p\widetilde{g'}(p) + g(0)\right]\widetilde{h_{1-\alpha}}(p) \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left[(\widetilde{g' * h_{1-\alpha}})(p) + g(0)\widetilde{h_{1-\alpha}}(p)\right].\end{aligned}$$

So

$$f(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\left[\int_0^t \frac{g'(s)}{(t-s)^{1-\alpha}}ds + \frac{g(0)}{t^{1-\alpha}}\right] = \frac{\sin(\pi\alpha)}{\pi}\left[\int_0^t \frac{g'(s)}{(t-s)^{1-\alpha}}ds + \frac{g(0)}{t^{1-\alpha}}\right].$$

We want now prove that

$$\boxed{\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}}$$

First of all, let us recall some properties of the gamma function. If we consider $B(\alpha,\beta) = \int_0^1 (1-x)^{\alpha-1}x^{\beta-1}dx$, the *beta function*, we have that

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha+\beta)B(\alpha,\beta).$$

Indeed, using the change of variables $t = x+y$, $p = y$ and $s = p/t$ at the right moment, we have that

$$\begin{aligned}\Gamma(\alpha)\Gamma(\beta) &= \left(\int_0^\infty e^{-x}x^{\alpha-1}dx\right)\left(\int_0^\infty e^{-y}y^{\beta-1}dy\right) = \int_0^\infty\int_0^\infty e^{-(x+y)}x^{\alpha-1}y^{\beta-1}dxdy \\ &= \int_0^\infty\int_0^t e^{-t}(t-p)^{\alpha-1}p^{\beta-1}\frac{dxdy}{dtdp}dtdp = \int_0^\infty e^{-t}\int_0^t (t-p)^{\alpha-1}p^{\beta-1}dtdp \\ &= \int_0^\infty e^{-t}t^{\alpha-1}\left(\int_0^t (1-\frac{p}{t})^{\alpha-1}p^{\beta-1}dp\right)dt \\ &= \int_0^\infty e^{-t}t^{\alpha+\beta-1}\left(\int_0^1 (1-s)^{\alpha-1}s^{\beta-1}ds\right)dt \\ &= \left(\int_0^\infty e^{-t}t^{\alpha+\beta-1}dt\right)\left(\int_0^1 (1-s)^{\alpha-1}s^{\beta-1}ds\right) \\ &= \Gamma(\alpha+\beta)B(\alpha,\beta).\end{aligned}$$

Hence, in our case, we have that $\Gamma(\alpha)\Gamma(1-\alpha) = \Gamma(1)B(\alpha,1-\alpha) = B(\alpha,1-\alpha)$.
We have only to compute the following integral

$$\int_0^\infty (1-x)^{-\alpha} x^{\alpha-1} dx.$$

Using the change of variable $t = \frac{x}{1-x}$, the problem reduces to solve another integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx \quad \text{with } 0 < \Re(\alpha) < 1.$$

To solve this integral we have to work in the complex plane, i.e. we have to consider the integral

$$\int_{\mathbb{R}^+} \frac{z^{\alpha-1}}{1+z} dz.$$

The problem is that the complex function $\frac{z^{\alpha-1}}{1+z}$ is a multi-valued function. To solve this problem (in Analysis), consider the cut complex plane $\mathbb{C} \setminus \{z \in \mathbb{C} : \Im(z) = 0 \text{ and } \Re(z) \geq 0\}$. If we take $x \in \{z \in \mathbb{C} : \Im(z) = 0 \text{ and} \Re(z) \geq 0\}$, we have that the determination from above of $x^{\alpha-1}$ is $x^{\alpha-1}$, while the determination from below of it is $\left(xe^{i2\pi}\right)^{\alpha-1}$. Hence we have a discontinuity between the values.
If we consider the determination from above our integral becomes

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x} dx,$$

that is the integral we are looking for; while, if we consider the determination from below of the integrand function we obtain the integral

$$e^{i2\pi(\alpha-1)} \int_0^\infty \frac{x^{\alpha-1}}{1+x} dx.$$

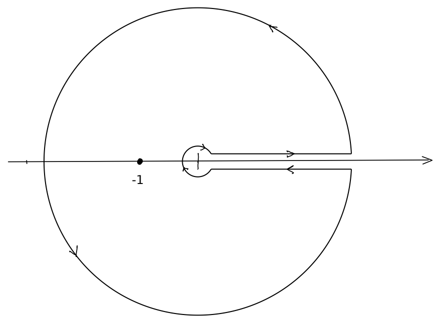
To solve our integral in complex plane we have to consider a circular path γ around the origin with radius $R > 1$ less a circular path around the origine with radius $\rho << 1$.

In this case we can not use the Cauchy theorem to state that the integral is zero because there is a singularity at the point -1. Hence, recalling that $e^{2\pi i} = 1$, we have

$$\begin{aligned}\int_\gamma \frac{z^{\alpha-1}}{1+z} dz = &\int_0^R \frac{x^{\alpha-1}}{1+x} dx + \int_0^{2\pi} \frac{(Re^{i\vartheta})^{\alpha-1}}{1+Re^{i\vartheta}} iRe^{i\vartheta} d\vartheta \\ &- \int_0^R \frac{x^{\alpha-1} e^{i2\pi(\alpha-1)}}{1+x} dx - \int_0^{2\pi} \frac{(\rho e^{i\vartheta})^{\alpha-1}}{1+\rho e^{i\vartheta}} i\rho e^{i\vartheta} d\vartheta = 2\pi i \text{Res}(-1)\end{aligned}$$

In this case we can find a simple expression for the residue simply considering a circular closed path around the point -1 with radius $\rho << 1$; hence we get

$$\int_0^{2\pi} \frac{(-1+\rho e^{i\vartheta})^{\alpha-1}}{1+(-1+\rho e^{i\vartheta})} i\rho e^{i\vartheta} d\vartheta = i \int_0^{2\pi} (\rho e^{i\vartheta} - 1)^{\alpha-1} d\vartheta \to 2\pi i (-1)^{\alpha-1} = 2\pi i \text{Res}(-1),$$



when ρ tends to zero.
Hence

$$\left[1-e^{i2\pi(\alpha-1)}\right]\int_0^R \frac{x^{\alpha-1}}{1+x}dx+I_1+I_2=2\pi i(-1)^{\alpha-1},$$

where

$$|I_1|=\left|\int_0^{2\pi}\frac{(Re^{i\vartheta})^{\alpha-1}}{1+Re^{i\vartheta}}iRe^{i\vartheta}d\vartheta\right|\leq\int_0^{2\pi}\frac{R^\alpha}{|1+Re^{i\vartheta}|}d\vartheta\leq 2\pi\frac{R^\alpha}{R-1}\ \to 0,$$

when R tends to infinite becuase $\alpha<1$, and

$$|I_2|=\left|\int_0^{2\pi}\frac{(\rho e^{i\vartheta})^{\alpha-1}}{1+\rho e^{i\vartheta}}i\rho e^{i\vartheta}d\vartheta\right|\leq\int_0^{2\pi}\frac{\rho^\alpha}{|1+\rho e^{i\vartheta}|}d\vartheta\ \to 0,$$

when ρ tends to zero.
Hence, passing to the limit for $R\to\infty$ and $\rho\to 0$, we have that

$$\left[1-e^{i2\pi(\alpha-1)}\right]\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx=2\pi i(-1)^{\alpha-1}.$$

Recalling that $(-1)^{\alpha-1}=-e^{i\pi\alpha}$ and that $e^{i2\pi(\alpha-1)}=e^{i2\pi\alpha}$, after some algebraic computation, we have that

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx=\frac{2\pi i(-1)^{\alpha-1}}{1-e^{i2\pi(\alpha-1)}}=\frac{\pi}{\sin(\pi\alpha)},$$

so we have proved our initial statement.

There is another method to compute the integral

$$\int_0^\infty \frac{x^{\alpha-1}}{1+x}dx$$

without using multi-values functions. Let us consider an $\alpha' \to \alpha$ wtih $0<\alpha,\alpha'<1$, hence

$$\int_0^\infty \frac{x^{\alpha'-1}}{1+x}dx \;\to\; \int_0^\infty \frac{x^{\alpha-1}}{1+x}dx.$$

So, we can consider a rational number m/n with $0<m/n<1$ such that $m/n\to\alpha$, compute the integral

$$\int_0^\infty \frac{x^{m/n-1}}{1+x}dx$$

and than pass to the limit for $m/n\to\alpha$. Hence our problem reduces to compute the integral

$$\int_0^\infty \frac{x^{m/n-1}}{1+x}dx.$$

Changing variable $\xi = x^{1/n}$, it remains to compute

$$n\int_0^\infty \frac{\xi^{m-1}}{1+\xi^n}d\xi.$$

Now we can pass to the complex plane, but, in this case, there are not problems of multi-values because the function is meromorphic; we have only a lot of singularity: all the roots of -1 and their number depends on n.

Exercise 17 *Solve the integral*

$$\int_0^\infty \frac{\xi^{m-1}}{1+\xi^n}d\xi.$$

*(*Hint: *consider a closed path with the form of a slice centred in zero, with radius* $R>1$ *and with generator straight line passed through the point* 1 *and the point* $e^{i\frac{2\pi}{n}}$ *rispectively)*
*(*Warning: *there is a singularity at the point* $e^{i\frac{\pi}{n}}$, *inside the path)*

Exercise 18 *Compute the same integral without using complex methods.*

3.2.3 Laplace transform and ODE

In general integral transforms are very useful tools when we want to solve linear problems like for example ordinary differential equation. Because ordinary differential equation of order n with n greater than 1 can be transform in a system of ordinary differential equation of the first order, we are interested to problems of this kind

$$\begin{cases} \dfrac{du}{dt} = Au \\ u(0) = u_0 \end{cases}$$

where $u : \mathbb{R} \to \mathbb{R}^n$ and A is a matrix $n \times n$.
We want to solve this problem. First of all, let us consider the exponential of a matrix e^{At}. It has a meaning as a Taylor series, hence for all matrix A we can define

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

It is easy to proof, using the definition as Taylor series, that, for all matrix A, we have

$$\frac{d}{dt} e^{At} = Ae^{At} = e^{At} A.$$

Using this statement we can prove that the solution of the initial problem is given by

$$u(t) = e^{At} u_0.$$

Indeed,

$$\left(\frac{du}{dt} - Au\right) = 0 \;\Rightarrow\; e^{-At}\left(\frac{du}{dt} - Au\right) = 0 \;\Rightarrow\; \frac{d}{dt}\left[e^{-At}u\right] = 0$$

$$\Rightarrow\; e^{-At}u(t) = C \;\Rightarrow\; u(t) = e^{At}C$$

Using the initial data we get that $C = u_0$, so we have that $u(t) = e^{At}u_0$. In this way we have proved the uniqueness, i.e., we have proved that, if the solution of the problem exists, it is unique and it is given by the expression above. To prove the existence we have only to check that $u(t) = e^{At}u_0$ is a solution.

Example 17. Consider the equation of an harmonic oscillator $u''(t) + \omega^2 u(t) = 0$. This is a linear ordinary differential equation of second order. Let us put $v(t) = u'(t)$ and $v'(t) = u''(t)$. If

$$w = \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$$

the problem reduces to $w' = Aw$, that is a problem of the same kind of our initial problem. Hence, its solution is $w(t) = Ce^{At}$ for some constant C. Let us compute the exponential of the matrix A. We have

$$\begin{aligned} e^{At} &= I + At - \frac{\omega^2 t^2}{2!} I - \frac{\omega^2 t^3}{3!} A + \dots \\ &= I\left(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \frac{\omega^6 t^6}{6!} + \dots\right) + A\left(t - \frac{\omega^2 t^3}{3!} + \frac{\omega^4 t^5}{5!} - \dots\right) \\ &= I\cos(\omega t) + \frac{A}{\omega}\sin(\omega t). \end{aligned}$$

Let us now consider the following problem

$$\begin{cases} \dfrac{du}{dt} = Au + f(t) \\ u(0) = u_0 \end{cases}$$

where $u : \mathbb{R} \to \mathbb{R}^n$, A is a matrix $n \times n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given function.
We have

$$\left(\frac{du}{dt} - Au\right) = f(t) \;\;\Rightarrow\;\; e^{-At}\left(\frac{du}{dt} - Au\right) = e^{-At} f(t) \;\;\Rightarrow\;\; \frac{d}{dt}\left[e^{-At}u\right] = e^{-At} f(t)$$

$$\Rightarrow\;\; e^{-At}u(t) - u_0 = \int_0^t e^{-As} f(s)ds \;\;\Rightarrow\;\; u(t) = e^{At}u_0 + \int_0^t e^{A(t-s} f(s)ds,$$

where $e^{At}u_0$ is the solution of the homogeneus equation and the integral is a particular solution of the problem. Also in this case we have proved the uniqueness, i.e., we have proved that, if the solution of the problem exists, it is unique and it is given by the expression above. To prove the existence we have only to check that this particular $u(t)$ is a solution.
We want now apply the Laplace transform on this problem. We are not sure that the Laplace transform of u exists, hence we will search for an $u \in L^1_{loc}(\mathbb{R}^+)$ such that $|u(t)| \leq Me^{\omega t}$, so the Laplace transform certainly exists.
First of all, let us observe that the matrix A is a linear operator indipendent of t, hence, because the Laplace transform is a linear operator, we have that

$$\mathscr{L}(Au) = A\widetilde{u},$$

where $\mathscr{L}$ indicate the Laplace operator. Hence our problem, after applying the Laplace transform, becomes

$$p\widetilde{u}(p) - u_0 = A\widetilde{u}(p) + \widetilde{f}(p) \quad\Rightarrow\quad (pI - A)\widetilde{u}(p) = u_0 + \widetilde{f}(p).$$

To find $\widetilde{u}$ we have to reverse $pI - A$. Is it possible? This is possible if and only if $\det(pI - A) \neq 0$, i.e., if and only if p is not an eigenvalue for the matrix A. The eigenvalues of the matrix A are n complex numbers, $\lambda_1, \ldots, \lambda_n$; the Laplace transform has a sense only if $\Re(p) > \omega$, hence if we choose $\omega > \max \Re(\lambda_i)$ it is possible to invert $pI - A$ without any problems. Hence

$$\widetilde{u}(p) = (pI - A)^{-1}u_0 + (pI - A)^{-1}\widetilde{f}(p)$$

Let us compare this statement with the expression of the solution found before

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s} f(s)ds.$$

It is easy to see that the integral is a convolution product $e^{At} * f$ and that $\mathscr{L}(e^{At}) = (pI - A)^{-1}$.

3.3 Inverse Laplace transform

Let us consider the function e^{zt} for $z \in \mathbb{C}$ fixed. If $z = \alpha \in \mathbb{R}$ we can apply to the function $e^{\alpha t}$ only the theory of the Laplace transforms; if $z = i\beta$ with $\beta \in \mathbb{R}$ we can apply to the function $e^{i\beta t}$ the theory of the Fourier series, the theory of the Fourier transforms in $\mathscr{S}'$ and the theory of the Laplace transforms; if $z = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$ we can apply to e^{zt} only the theory of Laplace transforms because the real part of the complex number z gives some problem.
We have that

$$\mathscr{L}(e^{zt})(p) = \int_0^\infty e^{zt} e^{-pt} dt = \frac{1}{p-z} \quad \text{with } \Re(p) > \Re(z).$$

Now, is $\frac{1}{p-z}$ the Laplace transform of some functions? The answer of this question is trivially yes because it is the Laplace transform of the function e^{zt}. But are we sure that does not exists another function having $\frac{1}{p-z}$ has Laplace transform? Also the answer of this question is yes because it can be proved an uniqueness theorem that state that if two functions have the same Laplace transforms, they are equal.
Let us suppose we have a Laplace transform. How can we reconstruct the beginnig function? We have to introduce the inverse Laplace transform. Let $f \in L^1_{loc}(R^+)$ such that $|f(t)| \le Me^{\omega t}$ and let us extend f to zero out of $\mathbb{R}^+$. Let $p = x + iy$. We have that

$$\begin{aligned}
\widetilde{f}(p) &= \int_0^\infty e^{-pt} f(t) dt = \int_{-\infty}^\infty e^{-pt} f(t) dt \\
&= \int_{-\infty}^\infty e^{-iyt} \left(e^{-xt} f(t)\right) dt = \widetilde{f}(x+iy) \\
&= \mathscr{F}\left(e^{-xt} f(t)\right).
\end{aligned}$$

Hence

$$e^{-xt} f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{iyt} \widetilde{f}(x+iy) dy \quad \Rightarrow \quad f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{(x+iy)t} \widetilde{f}(x+iy) dy.$$

We can see this last integral as an integral in the complex plane made on the path $x = \gamma$. Hence, if $p = \gamma + iy$, we finally have the inverse Laplace transform

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt} \widetilde{f}(p) dp.$$

It could be possible to prove an inversion theorem, i.e., we could prove that this is the inverse Laplace transform, and that using this definition $f(t) = 0$ for $t < 0$. We will prove the inversion theorem in a very particular case, i.e., in the case of $\widetilde{f}(p) = \frac{1}{p-z}$.
Let us consider the integral

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{pt}}{p-z}dp.$$

First of all, we want to check if this integral has a sense. We have that

$$\left|\frac{e^{pt}}{p-z}\right| = \left|\frac{e^{(\gamma+iy)t}}{\gamma+iy-z}\right| = \frac{e^{\gamma t}}{\sqrt{(\gamma-a)^2+(y-b)^2}} \approx \frac{1}{y} \notin L^1,$$

hence the function $\frac{e^{pt}}{p-z}$ is not integrable, so the integral above does not have a meaning. However it could have a sense as improper integral. To prove this we have to check if there exists the limit

$$\lim_{L\to\infty}\frac{1}{2\pi i}\int_{-L}^{L}\frac{e^{(\gamma+iy)t}}{\gamma+iy-z}dy.$$

To prove this we have to compute the integral in the complex plane using a closed path given by a segment long $2L$ passing for $x=\gamma$ and a semicircle C_R centred in zero with radius R.
Let us recall an important tool of complex analysis. Let f be an holomorphic function.

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{f(p)}{p-z}dp = f(z),$$

where γ is any closed path around the point z.
Using this tool of complex analysis, we have that our improper integral exists and is equal to e^{zt} on condition that the integral on C_R tends to zero when R goes to infinite. Hence we have only to prove that the integral on C_R goes to zero for $R\to\infty$. Using the Pitagora theorem and some trigonometric tools we have that

$$L=\sqrt{R^2-\gamma^2} \quad \text{and} \quad \vartheta_R = \arcsin\left(\frac{\sqrt{R^2-\gamma^2}}{R}\right),$$

where ϑ_R is the angle between the real axis and the segment from zero to L.
We have that

$$\left|\int_{C_R}\frac{e^{pt}}{p-z}dp\right| = \left|\int_{\vartheta_R}^{2\pi-\vartheta_R}\frac{e^{tRe^{i\vartheta}}}{Re^{i\vartheta}-z}iRe^{i\vartheta}d\vartheta\right| \le \int_{\vartheta_R}^{2\pi-\vartheta_R}\frac{e^{tR\cos(\vartheta)}}{|Re^{i\vartheta}-z|}Rd\vartheta.$$

We have to find the minimum of the distance $\left|Re^{i\vartheta}-z\right|$. Let us observe that this distance depends on z but certainly has the same order of R. Hence

$$\int_{\vartheta_R}^{2\pi-\vartheta_R}\frac{e^{tR\cos(\vartheta)}}{|Re^{i\vartheta}-z|}Rd\vartheta \approx \int_{\vartheta_R}^{2\pi-\vartheta_R}e^{tR\cos(\vartheta)}d\vartheta \le 2\int_{\vartheta_R}^{\frac{\pi}{2}}e^{tR\cos(\vartheta_R)}d\vartheta + \int_{\frac{\pi}{2}}^{\frac{3}{2}\pi}e^{tR\cos(\vartheta)}d\vartheta.$$

The second integral tends to zero when R tends to infinite because $\cos(\vartheta)\le 0$ when $\vartheta\in\left[\frac{\pi}{2},\frac{3}{2}\pi\right]$; let us observe the first one.

$$
\begin{aligned}
2\int_{\vartheta_R}^{\frac{\pi}{2}} e^{tR\cos(\vartheta_R)} d\vartheta &= C\int_{\vartheta_R}^{\frac{\pi}{2}} d\vartheta = C\left(\frac{\pi}{2} - \vartheta_R\right) \\
&= C\left(\frac{\pi}{2} - \arcsin\left(\frac{\sqrt{R^2-\gamma^2}}{R}\right)\right) \to C(\frac{\pi}{2} - \frac{\pi}{2}) = 0
\end{aligned}
$$

when R goes to infinite.
Hence we have proved that exists, as improper integral, the inverse Laplace transform of the function $\frac{1}{p-z}$ and that it is equal to e^{zt}.

Appendix A
Some integral computations

In this section we want to compute some integrals that could be useful.
By a direct computation we have that

$$\Big(e^{-\alpha x}\frac{-\alpha\cos(\beta x)+\beta\sin(\beta x)}{\alpha^2+\beta^2}\Big)' = e^{-\alpha x}\cos(\beta x)$$

and

$$\Big(-e^{-\alpha x}\frac{\alpha\sin(\beta x)+\beta\cos(\beta x)}{\alpha^2+\beta^2}\Big)' = e^{-\alpha x}\sin(\beta x)$$

hence, for $\alpha > 0$

$$\boxed{\begin{aligned}\int_0^\infty e^{-\alpha x}\cos(\beta x)dx &= \frac{\alpha}{\alpha^2+\beta^2}\\ \int_0^\infty e^{-\alpha x}\sin(\beta x)dx &= \frac{\beta}{\alpha^2+\beta^2}\end{aligned}}$$

• First of all we want to prove that

$$\boxed{\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}}$$

We have that

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \lim_{p\to\infty}\int_0^p \frac{\sin x}{x}dx. \tag{A.1}$$

Actually, integrating by parts, we have that

$$\int_0^p \frac{\sin^2 x}{x^2}dx = -\frac{1}{p}\sin^2 p + \int_0^{p/2}\frac{\sin x}{x}dx$$

from which the statement (A.1) passing to the limit.
Now, we have

$$\int_\varepsilon^A \frac{\sin x}{x}dx = \int_\varepsilon^A \sin x\Big(\int_0^\infty e^{-tx}dt\Big)dx = \int_0^\infty \Big(\int_\varepsilon^A e^{-tx}\sin x \mathrm{d}x\Big)dt$$

where we use the Fubini-Tonelli theorem to change the integrals.
Hence

$$\begin{aligned}\int_\varepsilon^A \frac{\sin x}{x}dx &= \int_0^\infty \Big(-e^{-tA}\frac{t\sin(A)+\cos(A)}{1+t^2} + e^{-t\varepsilon}\frac{t\sin(\varepsilon)+\cos(\varepsilon)}{1+t^2}\Big)dt \\ &= \int_0^\infty -e^{-tA}\frac{t\sin(A)+\cos(A)}{1+t^2}dt + \int_0^\infty e^{-t\varepsilon}\frac{t\sin(\varepsilon)}{1+t^2}dt \\ &+ \int_0^\infty e^{-t\varepsilon}\frac{\cos(\varepsilon)}{1+t^2}dt\end{aligned}$$

The module of the first integral is majorated

$$\Big|\int_0^\infty -e^{-tA}\frac{t\sin(A)+\cos(A)}{1+t^2}dt\Big| \le C\int_0^\infty e^{-tA}dt = \frac{C}{A} \to 0, \quad \text{for } A \to \infty$$

$$\frac{\sin(\varepsilon)}{\varepsilon}\sqrt{\varepsilon}\int_0^\infty e^{-t\varepsilon}\frac{\sqrt{t\varepsilon}\sqrt{t}}{1+t^2}dt \le C\sqrt{\varepsilon}\int_0^\infty \frac{\sqrt{t}}{1+t^2}dt \le C\sqrt{\varepsilon} \to 0, \quad \text{for } \varepsilon \to 0$$

Hence

$$\int_0^\infty \frac{\sin x}{x}dx = \int_0^\infty \frac{1}{1+t^2}dt = \frac{\pi}{2}.$$

We can use also the following strategy: let us consider the function

$$\varphi(t) = \int_0^\infty e^{-tx}\frac{\sin^2(x)}{x^2}dx, \qquad \varphi(\infty) = 0.$$

This function is 2-times derivable and they are given by

$$\varphi'(t) = -\int_0^\infty e^{-tx}\frac{\sin^2(x)}{x}dx, \qquad \varphi'(\infty) = 0$$

$$\varphi''(t) = \int_0^\infty e^{-tx}\sin^2(x)dx = \frac{1}{2}\int_0^\infty e^{-tx}(1-\cos(2x))dx.$$

This last integral can be easily computed (by using preceding formulas), that is

$$\varphi''(t) = \frac{1}{2}\Big(\frac{1}{t} - \frac{t}{4+t^2}\Big),$$

from which we get

$$\varphi'(t) = A + \frac{1}{2}\Big(\log t - \frac{1}{2}\log(4+t^2)\Big) = A + \frac{1}{2}\log\Big(\frac{t}{\sqrt{4+t^2}}\Big),$$

where $A = 0$ because $\varphi'(\infty) = 0$. Hence

$$\varphi'(t) = \frac{1}{2}\log\Big(\frac{t}{\sqrt{4+t^2}}\Big),$$

and the primitive is given by

$$\varphi(t) = B + \frac{1}{2}t\,\log\Big(\frac{t}{\sqrt{4+t^2}}\Big) - \arctan\Big(\frac{t}{2}\Big)$$

where $B = \frac{\pi}{2}$ because $\varphi(\infty) = 0$. Hence,

$$\varphi(t) = \frac{\pi}{2} + \frac{1}{2}t\,\log\Big(\frac{t}{\sqrt{4+t^2}}\Big) - \arctan\Big(\frac{t}{2}\Big),$$

from which we get

$$\int_0^\infty \frac{\sin^2(x)}{x^2}dx = \varphi(0) = \frac{\pi}{2}.$$

• We want now to prove that

$$\boxed{\int_{\mathbb{R}} e^{-t^2}dt = \sqrt{\pi}}$$

Let us consider the function $\phi(x)$

$$\phi(x) = \int_0^x e^{-t^2}dt,$$

which derivative is simply

$$\phi'(x) = e^{-x^2}.$$

Hence

$$(\phi(x)^2)' = 2\phi'(x)\phi(x) = 2\int_0^x e^{-(x^2+t^2)}dt = 2x\int_0^1 e^{-x^2(1+t^2)}dt$$

from which

$$(\phi(x)^2)' = -\Big(\int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}dt\Big)',$$

in other terms,

$$\phi(x)^2 + \int_0^1 \frac{e^{-x^2(1+t^2)}}{1+t^2}dt = \mathsf{constant} = \int_0^1 \frac{1}{1+t^2}dt = \frac{\pi}{4}$$

from which, for $x \to \infty$

$$\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}.$$

• We want to prove that

$$\boxed{\int_{\mathbb{R}} e^{itx} e^{-x^2}\,dx = \sqrt{\pi}\, e^{-\frac{t^2}{4}}}$$

We can first of all observe that we easily have that

$$\int_{\mathbb{R}} \sin(tx)\, e^{-x^2}\,dx = 0.$$

Hence, denoting by

$$\varphi(t) = \int_{\mathbb{R}} \cos(tx)\, e^{-x^2}\,dx,$$

we have that

$$\varphi'(t) = -\int_{\mathbb{R}} x\sin(tx) e^{-x^2}\,dx = \frac{1}{2}\int_{\mathbb{R}} \sin(tx) d\left(e^{-x^2}\right),$$

from which, integrating by parts, we obtain

$$\varphi'(t) = -\frac{t}{2}\varphi(t).$$

Integrating and noting that $\varphi(0) = \sqrt{\pi}$, we get the statement.

• We want to prove that

$$\boxed{\int_{\mathbb{R}} \frac{e^{itx}}{1+x^2}\,dx = \pi e^{-|t|}}$$

We have easily have that

$$\int_{\mathbb{R}} \frac{\sin(tx)}{1+x^2}\,dx = 0.$$

Hence we could write that

$$\int_{\mathbb{R}} \frac{\cos(tx)}{1+x^2}\,dx = \int_{\mathbb{R}} \cos(tx)\Big(\int_0^\infty e^{-s(1+x^2)}\,ds\Big)dx,$$

from which, changing the order of the integrals, we obtain

$$\int_{\mathbb{R}} \frac{\cos(tx)}{1+x^2}\,dx = \int_0^\infty e^{-s}\Big(\int_{\mathbb{R}} e^{-sx^2}\cos(tx)dx\Big)ds.$$

We have reduced ourselves to compute the integral

$$\psi(t) = \int_{\mathbb{R}} e^{-sx^2}\cos(tx)dx.$$

It is sufficient to observe that

$$\psi(t) = \frac{1}{\sqrt{s}}\varphi\Big(\frac{t}{\sqrt{s}}\Big),$$

hence

$$\psi(t) = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-\frac{t^2}{4s}}.$$

Summing up, we have that

$$\int_{\mathbb{R}} \frac{\cos(tx)}{1+x^2} dx = \sqrt{\pi} \int_0^\infty e^{-\left(s+\frac{t^2}{4s}\right)} \frac{1}{\sqrt{s}} ds.$$

To compute this last integral we observe that, using $r = \sqrt{s}$ and the identity

$$r^2 + \frac{t^2}{4r^2} = \left(r - \frac{|t|}{2r}\right)^2 + |t|$$

we have

$$\sqrt{\pi} \int_0^\infty e^{-\left(s+\frac{t^2}{4s}\right)} \frac{1}{\sqrt{s}} ds = 2\sqrt{\pi} e^{-|t|} \int_0^\infty e^{-\left(r-\frac{|t|}{2r}\right)^2} dr.$$

Using the variable $\xi = r - \frac{|t|}{2r}$, we obtain

$$\int_0^\infty e^{-\left(r-\frac{|t|}{2r}\right)^2} dr = \int_{\mathbb{R}} e^{-\xi^2} \left(\tfrac{1}{2} + \tfrac{\xi}{2\sqrt{\xi^2+2|t|}}\right) d\xi = \frac{\sqrt{\pi}}{2}$$

We can use also the following strategy: first of all we can observe that the function

$$\rho(t) = \int_0^\infty \frac{\cos(tx)}{1+x^2} dx = \int_0^\infty \frac{\cos(tx)}{1+x^2} dx$$

is a symmetrical function of t, hence we can consider only $t > 0$. Using the formulas deduced at the beginnig of the section, we have that

$$\begin{aligned}
\rho'(t) &= \frac{d}{dt} \int_0^\infty \frac{\cos(tx)}{1+x^2} dx = -\int_0^\infty \sin(tx) \frac{x}{1+x^2} dx \\
&= -\int_0^\infty \sin(tx) \Big[\int_0^\infty e^{-xz} \cos(z) dz \Big] dx \\
&= -\int_0^\infty \cos(z) \Big[\int_0^\infty e^{-xz} \sin(tx) dx \Big] dz \\
&= -\int_0^\infty \cos(z) \frac{t}{t^2+z^2} dz = -\int_0^\infty \cos(tx) \frac{t^2}{t^2+t^2x^2} dx \\
&= -\int_0^\infty \cos(tx) \frac{1}{1+x^2} dx = -\rho(t)
\end{aligned}$$

Summing up, we have obtained that $\rho'(t) = -\rho(t)$; hence integrating we have that

$$\rho(t) = e^{-t} \rho(0) = \frac{\pi}{2} e^{-t}.$$

Appendix B
Some useful Fourier series and integrals

In this section we want to recall some Fourier series and some integrals computed in the notes that can be useful.
Let us suppose $\alpha \notin \mathbb{Z}$. We have:

$$\cos(\alpha x) = \frac{2\sin(\alpha\pi)}{\pi}\Big[\frac{1}{2\alpha} + \sum_{k=1}^{\infty}(-1)^k \frac{\alpha}{\alpha^2 - k^2}\cos(kx)\Big]$$

$$\cosh(\alpha x) = \frac{2\sinh(\alpha\pi)}{\pi}\Big(\frac{1}{2\alpha} + \sum_{k=1}^{\infty}(-1)^k \frac{\alpha}{\alpha^2 + k^2}\cos(kx)\Big)$$

$$\sin(\alpha x) = \frac{2\sin(\alpha\pi)}{\pi}\sum_{k=1}^{\infty}(-1)^k \frac{k}{\alpha^2 - k^2}\sin(kx)$$

$$\sinh(\alpha x) = -\frac{2\sinh(\alpha\pi)}{\pi}\sum_{k=1}^{\infty}(-1)^k \frac{k}{\alpha^2 + k^2}\sin(kx)$$

$$e^{\alpha x} = \frac{2\sinh(\alpha\pi)}{\pi}\Big(\frac{1}{2\alpha} + \sum_{k=1}^{\infty}(-1)^k\Big[\frac{\alpha}{\alpha^2 + k^2}\cos(kx) - \frac{k}{\alpha^2 + k^2}\sin(kx)\Big]\Big)$$

Let us note some significant values of these Fourier series for $x = \pi$. We have:

$$\cos(\alpha\pi) = \frac{2\sin(\alpha\pi)}{\pi}\Big[\frac{1}{2\alpha} + \sum_{k=1}^{\infty}\frac{\alpha}{\alpha^2 - k^2}\Big]$$

$$\cos(\alpha\pi) = \frac{\sin(\alpha\pi)}{\pi}\Big[\frac{1}{\alpha} + \sum_{k=1}^{\infty}\frac{-\frac{2\alpha}{k^2}}{1 - \frac{\alpha^2}{k^2}}\Big]$$

$$\log(\sin(\alpha\pi)) = C + \log(\alpha) + \sum_{k=1}^{\infty}\log(1 - \frac{\alpha^2}{k^2})$$

$$\log(\sin(\alpha\pi)) = \log(\pi) + \log(\alpha) + \sum_{k=1}^{\infty}\log(1 - \frac{\alpha^2}{k^2})$$

$$\sin(\alpha\pi) = \pi\alpha \prod_{k=1}^{\infty}(1 - \frac{\alpha^2}{k^2})$$

$$\cosh(\alpha\pi) = \frac{2\sinh(\alpha\pi)}{\pi}\Big(\frac{1}{2\alpha} + \sum_{k=1}^{\infty} \frac{\alpha}{\alpha^2 + k^2}\Big)$$

$$\frac{\alpha\pi\coth(\alpha\pi) - 1}{2\alpha^2} = \sum_{k=1}^{\infty} \frac{1}{\alpha^2 + k^2}$$

We also have the following integrals:

$$\int_0^{\infty} e^{-\alpha x}\cos(\beta x)dx = \frac{\alpha}{\alpha^2 + \beta^2}$$

$$\int_0^{\infty} e^{-\alpha x}\sin(\beta x)dx = \frac{\beta}{\alpha^2 + \beta^2}$$

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$$

$$\int_{\mathbb{R}} e^{-t^2}dt = \sqrt{\pi}$$

$$\int_{\mathbb{R}} e^{itx}e^{-x^2}dx = \sqrt{\pi}e^{-\frac{t^2}{4}}$$

$$\int_{\mathbb{R}} \frac{e^{itx}}{1+x^2}dx = \pi e^{-|t|}$$

$$\int_0^{\infty} e^{-kx}\sin(\alpha x)dx = \frac{\alpha}{k^2 + \alpha^2} \quad k \in \mathbb{Z}$$

$$\int_0^{\infty} \frac{\sin(\alpha x)}{e^x - 1}dx = \frac{\pi\alpha\coth(\pi\alpha) - 1}{2\alpha}$$

$$\int_0^{\infty} \frac{x^{2k-1}}{e^x - 1}dx = \frac{(2\pi)^{2k}}{4k}B_{2k} \quad k \in \mathbb{Z}$$

$$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x}dx = \frac{\pi}{\sin(\pi\alpha)} \quad 0 < \alpha < 1$$

$$\int_0^{\infty} \frac{x^{m-1}}{1+x^n}dx = \frac{\pi}{n\sin(\pi\alpha)} \quad n,m \in \mathbb{Z} \quad 0 < \alpha < 1 \quad \text{s.t. } \frac{m}{n} \to \alpha$$

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