established that if $f \in M_w^2[0, \infty)$ and ζ_1, ζ_2 are bounded stopping times, $\zeta_1 \leqslant \zeta_2$, then

$$E\left\{ \int_{\zeta_1}^{\zeta_2} f(s)\, dw(s) | \mathcal{F}_{\zeta_1} \right\} = 0,$$

$$E\left\{ \left| \int_{\zeta_1}^{\zeta_2} f(s)\, dw(s) \right|^2 | \mathcal{F}_{\zeta_1} \right\} = E\left\{ \int_{\zeta_1}^{\zeta_2} |f|^2\, ds | \mathcal{F}_{\zeta_1} \right\}$$

a.s. The proof of these formulas is similar to the proof of Theorem 4.3. It employs Theorem 2.8 which remains valid for n-dimensional stochastic integrals.

We conclude this section with an extension of Theorem 6.5 to n dimensions.

Theorem 7.5. *Let $f = (f_1, \ldots, f_n)$ belong to $L_w^2[0, T]$, and let α, β be positive numbers. Then*

$$P\left\{ \max_{0 \leqslant t \leqslant T} \left[\int_0^t f(\lambda)\, dw(\lambda) - \frac{\alpha}{2} \int_0^t |f(\lambda)|^2\, d\lambda \right] > \beta \right\} \leqslant e^{-\alpha\beta}. \quad (7.12)$$

The proof is similar to the proof of Theorem 6.5. First we prove (7.12) in case $f(t)$ is a step function, using the martingale inequality, and then proceed to general f by approximation.

The inequality (7.12) is referred to as the *exponential martingale inequality*.

Corollary 6.6 also extends to the present n-dimensional case, i.e.,

$$\exp\left\{ \int_0^t f\, dw - \tfrac{1}{2} \int_0^t |f|^2\, ds \right\}$$

is a supermartingale.

PROBLEMS

1. Prove (2.20).
2. Prove Theorem 3.9 [*Hint*: Apply Theorem 3.6 to $\xi f(t)$, ξ bounded and $\mathcal{F}_\alpha$ measurable.]
3. Suppose $f \in L_w^2[0, \infty)$ and ζ is a stopping time such that $E \int_0^\zeta f^2(t)\, dt < \infty$. Prove that

$$E \int_0^\zeta f(t)\, dw(t) = 0, \qquad E \left| \int_0^\zeta f(t)\, dw(t) \right|^2 = E \int_0^\zeta f^2(t)\, dt.$$

4. Let

$$\rho(x) = \begin{cases} c \exp[1/(|x|^2 - 1)] & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

for $x \in R^n$, where c is a positive constant such that $\int_{R^n} \rho(x)\, dx = 1$. If f is a function locally integrable, then

$$(J_\epsilon f)(x) = \frac{1}{\epsilon^n} \int_{R^n} \rho\left(\frac{x-y}{\epsilon} \right) f(y)\, dy$$

is called a *mollifier* of f. Prove:

(i) $J_\epsilon f$ is in $C^\infty(R^n)$;

(ii) If K is a compact set and Ω a bounded open set containing K, then

$$(J_\epsilon f)(x) = \frac{1}{\epsilon^n} \int_{\Omega} \rho\left(\frac{x-y}{\epsilon} \right) f(y)\, dy$$

$$= \int_{|z|<1} \rho(z) f(x - \epsilon z)\, dz \qquad (x \in K),$$

provided $\epsilon < \operatorname{dist}(K, R^n \backslash \Omega)$.

(iii) If $f \in L^p(\Omega)$ for some $p \geqslant 1$, then

$$\left\{ \int_K |J_\epsilon f|^p\, dx \right\}^{1/p} \leqslant \left\{ \int_\Omega |f|^p\, dx \right\}^{1/p}.$$

(iv) If $f \in L^p(\Omega)$ for some $p \geqslant 1$, then

$$\int_K |J_\epsilon f - f|^p\, dx \to 0 \qquad \text{if} \quad \epsilon \to 0.$$

5. Let $f(x)$ be a continuous function for $\alpha \leqslant x \leqslant \beta$, and let

$$(P_k f)(x) = \frac{\int_\alpha^\beta \left[1 - (x-y)^2\right]^k f(y)\, dy}{\int_{-1}^1 (1 - y^2)^k\, dy} \qquad (k = 1, 2, \ldots).$$

Let δ be any positive number. Prove that $(P_k f)(x) \to f(x)$ uniformly in $x \in [\alpha + \delta, \beta - \delta]$ as $k \to \infty$. [*Hint*: $[\int_\epsilon^1 (1 - y^2)^k\, dy / \int_0^1 (1 - y^2)^k\, dy] \to 0$ if $k \to \infty$, for any $\epsilon > 0$.]

6. Let $f(x)$ be a continuous function in an n-dimensional interval $I \equiv \{x;\ \alpha_i \leqslant x \leqslant \beta_i,\ 1 \leqslant i \leqslant n\}$, and let

$$(P_k f)(x) = \frac{\int_{\alpha_1}^{\beta_1} \cdots \int_{\alpha_n}^{\beta_n} \prod_{i=1}^n \left[1 - (x_i - y_i)^2\right]^k f(y)\, dy_n \cdots dy_1}{\left[\int_{-1}^1 (1 - y^2)^k\, dy\right]^n}$$

$$(k = 1, 2, \ldots).$$

Let I_0 be any subset lying in the interior of I. Prove that, as $k \to \infty$,

$$(P_k f)(x) \to f(x) \quad \text{uniformly in} \quad x \in I_0.$$

Notice that $P_k f$ is a polynomial. It is called a ***polynomial mollifier*** of f.

7. If in the preceding problem f belongs to $C^m(I)$ and f vanishes in a neighborhood of the boundary of I, then

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}(P_kf)(x)\to\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}f(x)\qquad\text{if}\quad k\to\infty,$$

uniformly in $x\in I_0$, for any $(i_1,\ldots,i_n)$ such that $0\leqslant i_1+\cdots+i_n\leqslant m$.

8. If $f\in C^m(R^n)$, then there exists a sequence of polynomials Q_k such that, as $k\to\infty$,

$$\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}Q_k(x)\to\frac{\partial^{i_1+\cdots+i_n}}{\partial x_1^{i_1}\cdots\partial x_n^{i_n}}f(x)\qquad\text{for}\quad 0\leqslant i_1+\cdots+i_n\leqslant m,$$

uniformly in x in compact subsets of R^n. [*Hint*: Approximate f by functions with compact support, and apply Problem 7 to these functions.]

9. If in the previous problem it is assumed that f, f_{x_i} $(1\leqslant i\leqslant n)$ and $f_{x_ix_j}$ $(2\leqslant i,j\leqslant n)$ are continuous in R^n (instead of $f\in C^m(R^n)$), then

$$Q_k\to f,\qquad\frac{\partial}{\partial x_i}Q_k\to\frac{\partial f}{\partial x_i}\qquad(1\leqslant i\leqslant n),$$

$$\frac{\partial^2}{\partial x_i\,\partial x_j}Q_k\to\frac{\partial^2 f}{\partial x_i\,\partial x_j}\qquad(2\leqslant i,j\leqslant n)$$

uniformly on compact subsets of R^n.

10. Let $f(x,t)$ be a continuous function in $(x,t)\in\mathrm{R}^n\times[0,\infty)$ together with its derivatives f_t, f_{x_i}, $f_{x_ix_j}$. Prove that there exists a function F continuous in $(x,t)\in R^n\times R^1$ together with its derivatives F_t, F_{x_i}, $F_{x_ix_j}$, such that $F(x,t)=f(x,t)$ if $x\in R^n$, $t\geqslant 0$.

11. Let $f(x,t)=f(x_1,\ldots,x_n,t)$ be a continuous function in $(x,t)\in R^n\times[0,\infty)$ together with its derivatives f_t, f_{x_i}, $f_{x_ix_j}$. Then there exists a sequence of polynomials $Q_m(x,t)$ such that, as $m\to\infty$,

$$Q_m\to f,\qquad\frac{\partial}{\partial t}Q_m\to f_t,\qquad\frac{\partial}{\partial x_i}Q_m\to f_{x_i},\qquad\frac{\partial^2}{\partial x_i\,\partial x_j}Q_m\to f_{x_ix_j}$$

uniformly in compact subsets. [*Hint*: Combine Problems 9, 10.]

12. Prove (5.8).

13. Prove (5.14) and complete the proof of (5.13).

14. Let $f\in L_w^2[0,\infty)$, $|f|\leqslant K$ (K constant) and let $d\xi(t)=f(t)\,dw(t)$, $\xi(0)=0$ where $w(t)$ is a Brownian motion. Prove:

(i) if $f\leqslant\beta$, then $E|\xi(t)|^2\leqslant\beta^2t$;
(ii) if $f\geqslant\alpha>0$, then $E|\xi(t)|^2\geqslant\alpha^2t$.

15. Prove Theorem 5.3. [*Hint*: Proceed as in the proof of Theorem 5.2, but with

$$\Phi(w(t),t)=f(\xi_{10}+a_1t+b_1w(t),\ldots,\xi_{m0}+a_mt+b_mw(t))$$

where ξ_{i0}, a_i are random variables and the b_i are random n-vectors; cf. Step 4.]

16. Let $\xi(t) = \int_0^t b(t)\,dw(t)$ where b is an $n \times n$ matrix belonging to $L_w^2[0, \infty)$. Suppose that $d\xi_i\,d\xi_j = 0$ if $i \neq j$, $d\xi_i\,d\xi_i = dt$ (see (7.8) for the definition of $d\xi_i\,d\xi_j$), for all $1 \leqslant i, j \leqslant n$. Prove that $\xi(t)$ is an n-dimensional Brownian motion. [*Hint*: First proof: Use Theorem 3.6.2. Second proof: Suppose the elements of b are bounded step functions and let $\zeta(t) = \exp[i\gamma \cdot \xi(t) + \gamma^2 t/2]$. By Itô's formula $d\zeta = i\zeta\gamma\,dw$. By Theorem 2.8

$$E[e^{i\gamma \cdot \xi(t)} | \mathcal{F}_s] = e^{i\gamma \cdot \xi(s)} e^{-\gamma^2 (t-s)/2}.$$

Use Problem 2, Chapter 3.]

17. Let $\gamma > 0$, $a > 0$, $\tau = \min\{t;\, w(t) = a\}$ where $w(t)$ is one-dimensional Brownian motion. Prove that $P(\tau < \infty) = 1$ and

$$Ee^{-\gamma\tau} = \exp\left(-\sqrt{2\gamma}\, a\right).$$

[*Hint*: For any $c > 0$,

$$P\Big[\max_{0 \leqslant s \leqslant t} w(s) > c\Big] \leqslant P\Big[\max_{0 \leqslant s \leqslant t}\Big(w(s) - \frac{\alpha}{2} s\Big) > \beta\Big] < e^{-c^2/2t}$$

where $\alpha = c/t$, $\beta = c/2$. Hence $P(\tau < \infty) = 1$. Since $y(t) = \exp[\gamma w(t) - \gamma^2 t/2]$ is a martingale, so is $y(t \wedge \tau)$. Hence

$$E \exp\big[\gamma w(t \wedge \tau) - \tfrac{1}{2}\gamma^2 (t \wedge \tau)\big] = 1.$$

Take $t \uparrow \infty$.]

18. Under the conditions of the previous problem

$$P(\tau \in dt) = \frac{a}{(2\pi t^3)^{1/2}} \exp\left(-\frac{a^2}{2t}\right) dt.$$

[*Hint*: Use the fact (see, for instance, Feller [1]) that if the Laplace transform of two probability distributions concentrated on $[0, \infty)$ coincide, then the probability distributions coincide.]

19. If $w(t)$ is a Brownian motion and $0 \leqslant y$, $x < y$, then

$$P\Big(w(t) \in dx, \max_{0 \leqslant s \leqslant t} w(s) \in dy\Big)$$
$$= \left(\frac{2}{\pi t^3}\right)^{1/2} (2y - x) \exp\left[-\frac{(2y-x)^2}{2t}\right] dx\,dy.$$

[*Hint*: Use Problem 12, Chapter 2 and Theorem 3.6.3 to deduce that

$$P\Big[w(t) \in dx, \max_{0 \leqslant s \leqslant t} w(s) \geqslant y\Big] = \int_0^t P(\tau \in ds) P[w(t-s) + y \in dx]$$

where $\tau = \min\{t;\, w(t) = y\}$, and apply the preceding problem.]

20. Let $f(t)$ be a continuous process in $L_w^2[0, T]$ and let $\Pi_n : t_{n,0} = 0 < t_{n,1} < \cdots < t_{n,n} = T$ be a partition with mesh $|\Pi_n| \to 0$ as $n \to \infty$. Define

$$g_n(t) = \sum_{i=0}^{j-1} f(t_{n,i})(w(t_{n,i+1}) - w(t_{n,i})) + f(t_{n,j})(t - t_{n,j})$$
$$\text{if} \quad t_{n,j} \leqslant t < t_{n,j+1}.$$

Prove that for some subsequence $\{n'\}$ of $\{n\}$,

$$\sup_{0 \leqslant t \leqslant T} \left| \int_0^t f(s)\, dw(s) - g_{n'}(t) \right| \to 0 \quad \text{a.s.} \qquad \text{if} \quad n' \to \infty.$$

21. Let $\sigma(x, t)$ be a measurable function in $(x, t) \in R^n$ such that

$$|\sigma(x, t) - \sigma(\bar{x}, t)| \leqslant \eta(|x - \bar{x}|), \qquad \eta(\delta) \downarrow 0 \quad \text{if } \delta \downarrow 0,$$

and let $f(t)$ be an n-dimensional continuous process in $L_w^2[0, T]$. Let

$$\sigma_\epsilon(x, t) = \frac{1}{\epsilon} \int_{-1}^{T} \rho\left(\frac{t - s - \epsilon}{\epsilon} \right) \sigma(x, s)\, ds \qquad (2\epsilon < 1)$$

where $\rho(t)$ is defined as in Lemma 1.1 and $\sigma(x, s) = \sigma(x, 0)$ if $-1 < s < 0$. Prove:

(i) $\int_0^T |\sigma(x, t) - \sigma_\epsilon(x, t)|^2\, dt \to 0$ uniformly in x in bounded sets, as $\epsilon \to 0$.

(ii) $\int_0^T |\sigma(f(t), t) - \sigma_\epsilon(f(t), t)|^2\, dt \to 0$ a.s. as $\epsilon \to 0$.

(iii) $\sup_{0 \leqslant t \leqslant T} |\int_0^t \sigma(f(s), s)\, dw(s) - \int_0^t \sigma_{\epsilon_n}(f(s), s)\, dw(s)| \to 0$ a.s. for some sequence $\epsilon_n \downarrow 0$.

[*Hint*: for (i), use the uniform continuity in x of $\int \sigma(x, t)\, dt$ and of $\int \sigma_\epsilon(x, t)\, dt$.]