Sia W_t un processo di Wiener e $\{\mathcal{F}_t\}_{t\geq 0}$ una filtrazione *non anticipante*. Sia Inoltre X_t un processo a traiettorie C^1, adattato alla filtrazione tale che $(t,\omega) \to \dot{X}_t(\omega) \in L^2([0,T]\times\Omega)$.

Esiste l'integrale (alla Stieltjes) rispetto al processo di Wiener (traiettoria per traiettoria) ed è

$$\boxed{\int_0^t X_r\, dW_r = X_0\, W_t + \int_0^t (W_t - W_r)\, \dot{X}_r\, dr}$$

Teorema 1. *Risulta inoltre che*

$$\mathbb{E}\Big(\int_0^t X_r\, dW_r\Big) = 0$$

e

$$\mathbb{E}\Big[\Big(\int_0^t X_r\, dW_r\Big)^2\Big] = \mathbb{E}\int_0^t X_r^2\, dr.$$

Si può dimostrare di più; infatti

$$\mathbb{E}\Big(\int_0^t X_r\, dW_r \mid \mathcal{F}_s\Big) = \int_0^s X_r\, dW_r$$

e

$$\mathbb{E}\Big[\Big(\int_s^t X_r\, dW_r\Big)^2 \mid \mathcal{F}_s\Big] = \mathbb{E}\Big[\int_s^t X_r^2\, dr \mid \mathcal{F}_s\Big].$$

Dimostrazione. La prima è facile. Per la seconda consideriamo

$$\Big(X_0\, W_t + \int_0^t (W_t - W_r)\, \dot{X}_r\, dr\Big)^2 =$$
$$X_0^2\, W_t^2 + 2X_0\, W_t \int_0^t (W_t - W_r)\, \dot{X}_r\, dr + \int_0^t \int_0^t (W_t - W_r)\, \dot{X}_r (W_t - W_\rho)\, \dot{X}_\rho\, dr\, d\rho$$

dove il termine medio è conveniente riscriverlo

$$X_0\, W_t \int_0^t (W_t - W_r)\, \dot{X}_r\, dr = X_0\, \Big[\int_0^t (W_t - W_r)^2\, \dot{X}_r\, dr + \int_0^t W_r\, (W_t - W_r)\, \dot{X}_r\, dr\Big]$$

il cui valore di aspettazione risulta

$$\mathbb{E}\big[X_0 \int_0^t (t - r)\, \dot{X}_r\, dr\big] = \mathbb{E}\big[-t\, X_0^2 + X_0 \int_0^t X_r\, dr\big]$$

Il valore di aspettazione del primo più il secondo termine è quindi

$$t\,\mathbb{E}(X_0^2)+2\,\mathbb{E}\big[-t\,X_0^2+X_0\int_0^t X_r\,dr\big] = -t\,\mathbb{E}(X_0^2)+2X_0\int_0^t X_r\,dr = \int_0^t X_r^2\,dr-\int_0^t (X_r-X_0)^2\,dr$$

D'altra parte conviene riscrivere l'ultimo addendo (supponendo $r<\rho$)

$$(W_t-W_r)\,\dot X_r(W_t-W_\rho)\,\dot X_\rho = \dot X_r(W_t-W_\rho)^2\,\dot X_\rho + (W_\rho-W_r)\,\dot X_r(W_t-W_\rho)\,\dot X_\rho$$

da cui il valore di aspettazione risulta

$$\mathbb{E}(\dot X_r\,\dot X_\rho)(t-\rho)$$

e quindi

$$\mathbb{E}\bigg[\int_0^t\int_0^t (W_t-W_r)\,\dot X_r(W_t-W_\rho)\,\dot X_\rho\,dr\,d\rho\bigg] = 2\,\mathbb{E}\bigg[\int_0^t d\rho\int_0^\rho (W_t-W_r)\,\dot X_r(W_t-W_\rho)\,\dot X_\rho\,dr\bigg]$$

$$\mathbb{E}\bigg[\int_0^t\int_0^t (W_t-W_r)\,\dot X_r(W_t-W_\rho)\,\dot X_\rho\,dr\,d\rho\bigg] = 2\int_0^t d\rho\int_0^\rho \mathbb{E}(\dot X_r\,\dot X_\rho)(t-\rho)\,dr$$

$$= 2\mathbb{E}\int_0^t (t-\rho)\,\dot X_\rho d\rho\int_0^\rho \dot X_r\,dr = 2\mathbb{E}\int_0^t (t-\rho)\,\dot X_\rho(X_\rho-X_0)d\rho = \mathbb{E}\int_0^t (t-\rho)\,\frac{d}{d\rho}(X_\rho-X_0)^2 d\rho$$

$$= \mathbb{E}\Big[(t-\rho)(X_\rho-X_0)^2\Big]_0^t - \int_0^t (-1)(X_\rho-X_0)^2 d\rho$$

$$= \mathbb{E}\int_0^t (X_\rho-X_0)^2 d\rho$$

Sommando otteniamo

$$\mathbb{E}\Big[\Big(\int_0^t X_r\,dW_r\Big)^2\Big] = \mathbb{E}\int_0^t X_r^2\,dr.$$

In quanto alla terza proprietà, dalla

$$\boxed{\int_s^t X_r\,dW_r = X_s(W_t-W_s) + \int_s^t (W_t-W_r)\,\dot X_r\,dr}$$

otteniamo

$$\mathbb{E}\Big(\int_0^t X_r\,dW_r \mid \mathcal{F}_s\Big) = \mathbb{E}\Big(\int_s^t X_r\,dW_r \mid \mathcal{F}_s\Big) + \int_0^s X_r\,dW_r$$

$$\mathbb{E}\Big(\int_s^t X_r\,dW_r \mid \mathcal{F}_s\Big) = \mathbb{E}\Big(X_s(W_t-W_s) + \int_s^t (W_t-W_r)\,\dot X_r\,dr \mid \mathcal{F}_s\Big)$$

$$= X_s \mathbb{E}(W_t - W_s) + \mathbb{E}\Big(\int_s^t (W_t - W_r)\, \dot{X}_r \, dr \mid \mathcal{F}_s \Big)$$

$$= 0 + \int_s^t \mathbb{E}\Big(\mathbb{E}\big((W_t - W_r)\, \dot{X}_r \mid \mathcal{F}_r \big) \mid \mathcal{F}_s \Big)\, dr$$

$$= \int_s^t \mathbb{E}\Big(X_r \mathbb{E}(W_t - W_r) \mid \mathcal{F}_s \Big)\, dr = 0$$

Infine per dimostrare l'ultima proprietà si procede come per dimostrare la seconda

$$\mathbb{E}\Big[\Big(\int_0^t X_r \, dW_r \Big)^2 \mid \mathcal{F}_s \Big] = \int_0^s X_r^2 \, dr + 2 \int_0^s X_r \, dr \; \mathbb{E}\Big[\int_s^t X_r \, dW_r \mid \mathcal{F}_s \Big] + \mathbb{E}\Big[\Big(\int_s^t X_r \, dW_r \Big)^2 \mid \mathcal{F}_s \Big]$$

$$= \int_0^s X_r^2 \, dr + \mathbb{E}\Big[\Big(\int_s^t X_r \, dW_r \Big)^2 \mid \mathcal{F}_s \Big]$$

perché il termine medio è nullo per la terza formula. Calcoliamo l'ultimo addendo:

$$\Big(X_s (W_t - W_s) + \int_s^t (W_t - W_r)\, \dot{X}_r \, dr \Big)^2 =$$

$$X_s^2 (W_t - W_s)^2 + 2 X_s (W_t - W_s) \cdot \int_s^t (W_t - W_r)\, \dot{X}_r \, dr + \Big(\int_s^t (W_t - W_r)\, \dot{X}_r \, dr \Big)^2$$

Abbiamo tre termini. Per il primo

$$\mathbb{E}(X_s^2 (W_t - W_s)^2 \mid \mathcal{F}_s) = X_s^2 \, (t - s)$$

Per il secondo

$$X_s (W_t - W_s) \cdot \int_s^t (W_t - W_r)\, \dot{X}_r \, dr = X_s \int_s^t (W_t - W_r)^2 \, \dot{X}_r \, dr + X_s \int_s^t (W_r - W_s)(W_t - W_r)\, \dot{X}_r \, dr$$

$$\int_s^t (t - r)\, \dot{X}_r \, dr = -(t - s) X_s + \int_s^t X_r \, dr$$

e quindi sommando i primi due termini abbiamo

$$\mathbb{E}\Big[X_s^2 (W_t - W_s)^2 + 2 X_s (W_t - W_s) \cdot \int_s^t (W_t - W_r)\, \dot{X}_r \, dr \mid \mathcal{F}_s \Big] =$$

$$= -(t - s) X_s^2 + 2 X_s \int_s^t \mathbb{E}(X_r \mid \mathcal{F}_s)\, dr = \int_s^t \mathbb{E}((2 X_s X_r - X_s^2) \mid \mathcal{F}_s)\, dr$$

$$= \int_s^t \mathbb{E}(X_r^2 \mid \mathcal{F}_s)\, dr - \int_s^t \mathbb{E}((X_r - X_s)^2 \mid \mathcal{F}_s)\, dr$$

Infine per l'ultimo termine

$$\Big(\int_s^t (W_t - W_r)\, \dot{X}_r\, dr\Big)^2 = 2\int_s^t d\rho \int_s^\rho (W_t - W_r)\, \dot{X}_r (W_t - W_\rho)\, \dot{X}_\rho\, dr$$

$$\mathbb{E}(\dot{X}_r\, \dot{X}_\rho \mid \mathcal{F}_s)(t-\rho)$$

$$\mathbb{E}\Big(\Big(\int_s^t (W_t - W_r)\, \dot{X}_r\, dr\Big)^2 \mid \mathcal{F}_s\Big) = 2\int_s^t d\rho \int_s^\rho \mathbb{E}(\dot{X}_r\, \dot{X}_\rho \mid \mathcal{F}_s)(t-\rho)\, dr$$

$$= 2\mathbb{E}\Big(\int_s^t d\rho\, \dot{X}_\rho (t-\rho)(X_\rho - X_s) \mid \mathcal{F}_s\Big) = \mathbb{E}\Big(\int_s^t (t-\rho)\frac{d}{d\rho}(X_\rho - X_s)^2\, d\rho \mid \mathcal{F}_s\Big)$$

$$= \mathbb{E}\Big(\int_s^t (X_\rho - X_s)^2\, d\rho \mid \mathcal{F}_s\Big)$$

Il teorema 1 è dimostrato.

Inoltre abbiamo

$$\boxed{\mathbb{E}\Big[\Big(\int_0^t X_r\, dW_r\Big)^2 - \int_0^t X_r^2\, dr \mid \mathcal{F}_s\Big] = \Big(\int_0^s X_r\, dW_r\Big)^2 - \int_0^s X_r^2\, dr}$$

Infatti

$$\mathbb{E}\Big[\Big(\int_0^t X_r\, dW_r\Big)^2 - \int_0^t X_r^2\, dr \mid \mathcal{F}_s\Big] = \Big(\int_0^s X_r\, dW_r\Big)^2 - \int_0^s X_r^2\, dr$$

$$+\mathbb{E}\Big[\Big(\int_s^t X_r\, dW_r\Big)^2 + 2\int_0^s X_r\, dW_r \int_s^t X_r\, dW_r - \int_s^t X_r^2\, dr \mid \mathcal{F}_s\Big]$$

$$= \Big(\int_0^s X_r\, dW_r\Big)^2 - \int_0^s X_r^2\, dr$$