È facile vedere che la (6.10) implica

$$\Big(\int_\alpha^\beta X_n(t)^2\,dt\Big)^{\frac12} \underset{n\to\infty}{\overset{P}{\to}} \Big(\int_\alpha^\beta X(t)^2\,dt\Big)^{\frac12}$$

Infatti dalla disuguaglianza $|\|x\|-\|y\||\le\|x-y\|$ si ha

$$\mathbb{P}\Big(\Big|\Big(\int_\alpha^\beta X(t)^2\,dt\Big)^{\frac12}-\Big(\int_\alpha^\beta X_n(t)^2\,dt\Big)^{\frac12}\Big|>\varepsilon\Big)$$

$$\le\mathbb{P}\Big(\Big(\int_\alpha^\beta (X(t)-X_n(t))^2\,dt\Big)^{\frac12}>\varepsilon\Big)=\mathbb{P}\Big(\int_\alpha^\beta (X(t)-X_n(t))^2\,dt>\varepsilon^2\Big)\to 0$$

Dunque se $\rho'>\rho$, $\varepsilon'<\varepsilon$, abbiamo

$$\mathbb{P}\Big(\Big|\int_\alpha^\beta X(t)\,dB_t\Big|\ge\varepsilon\Big)\le\mathbb{P}\Big(\Big|\int_\alpha^\beta X_n(t)\,dB_t\Big|\ge\varepsilon'\Big)+\mathbb{P}\Big(\Big|\int_\alpha^\beta X(t)\,dB_t-\int_\alpha^\beta X_n(t))\,dB_t\Big|\ge\varepsilon-\varepsilon'\Big)$$

$$\mathbb{P}\Big(\int_\alpha^\beta X_n(t)^2\,dt\ge\rho'\Big)=\mathbb{P}\Big(\Big(\int_\alpha^\beta X_n(t)^2\,dt\Big)^{\frac12}\ge\sqrt{\rho'}\Big)$$

$$\le\mathbb{P}\Big(\Big(\int_\alpha^\beta X(t)^2\,dt\Big)^{\frac12}\ge\sqrt{\rho}\Big)+\mathbb{P}\Big(\Big|\Big(\int_\alpha^\beta X_n(t)\,dt\Big)^{\frac12}-\Big(\int_\alpha^\beta X(t)^2\,dt\Big)^{\frac12}\Big|\ge\sqrt{\rho'}-\sqrt{\rho}\Big)$$

e quindi

$$\mathbb{P}\Big(\int_\alpha^\beta X_n(t)^2\,dt\ge\rho'\Big)\le$$

$$\mathbb{P}\Big(\int_\alpha^\beta X(t)^2\,dt\ge\rho\Big)+\mathbb{P}\Big(\Big|\Big(\int_\alpha^\beta X_n(t)\,dt\Big)^{\frac12}-\Big(\int_\alpha^\beta X(t)^2\,dt\Big)^{\frac12}\Big|\ge\sqrt{\rho'}-\sqrt{\rho}\Big)$$

da cui per il Lemma 6.6

$$\mathbb{P}\Big(\Big|\int_\alpha^\beta X(t)\,dB_t\Big|\ge\varepsilon\Big)\le$$

$$\mathbb{P}\Big(\Big|\int_\alpha^\beta X_n(t)\,dB_t\Big|\ge\varepsilon'\Big)+\mathbb{P}\Big(\Big|\int_\alpha^\beta X(t)\,dB_t-\int_\alpha^\beta X_n(t))\,dB_t\Big|\ge\varepsilon-\varepsilon'\Big)$$

$$\le\mathbb{P}\Big(\int_\alpha^\beta X_n(t)^2\,dt\ge\rho'\Big)+\frac{\rho'}{\varepsilon'^2}+\mathbb{P}\Big(\Big|\int_\alpha^\beta X(t)\,dB_t-\int_\alpha^\beta X_n(t))\,dB_t\Big|\ge\varepsilon-\varepsilon'\Big)$$

$$\le\mathbb{P}\Big(\int_\alpha^\beta X(t)^2\,dt\ge\rho\Big)+\frac{\rho'}{\varepsilon'^2}$$

$$+\mathbb{P}\Big(\Big|\int_\alpha^\beta X(t)\,dB_t-\int_\alpha^\beta X_n(t))\,dB_t\Big|\ge\varepsilon-\varepsilon'\Big)$$

$$+\mathbb{P}\Big(\Big|\Big(\int_\alpha^\beta X_n(t)\,dt\Big)^{\frac12}-\Big(\int_\alpha^\beta X(t)^2\,dt\Big)^{\frac12}\Big|\ge\sqrt{\rho'}-\sqrt{\rho}\Big)$$

passando al limite per $n\to\infty$ e per l'arbitrarietà di $\rho'$, $\varepsilon'$ si ha la tesi.